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VOLUME XVII., NUMBER 1

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CONTENTS

PAGE

Note on Implicit Functions Defined by Two Equations when the Functional Determinant Vanishes. By Professor W. R. LONGLEY - - - - - - - - - - 1

Sturm's Method of Integrating $dx/\sqrt{X} + dy/\sqrt{Y}$. By Professor F. H. SAFFORD - - - - - - - - - 9

A Property of a Special Linear Substitution. By Professor F. R. SHARPE - - - - - - - - - - 15

On the Factorization of Integral Functions with p-adic Coefficients. By Professor L. E. DICKSON - - - - 19

Hensel's Theory of Algebraic Numbers. By Professor L. E. DICKSON - - - - - - - - - - - 23

Shorter Notices - - - - - - - - - - - 36

Notes - - - - - - - - - - - - - 47

New Publications - - - - - - - - - - 55

VOLUME XVII., NUMBER 2

NOVEMBER, 1910

Whole No. 192

CONTENTS

PAGE

The Seventeenth Summer Meeting of the American Mathematical Society. By Professor F. N. COLE - - - - 61

The Preparation of College and University Instructors in Mathematics. Committee Report - - - - - 77

Vector Analysis. By Dr. H. B. PHILLIPS - - - - 100

Notes - - - - - - - - - - - - 104

New Publications - - - - - - - - - - 112

VOLUME XVII., NUMBER 3

DECEMBER, 1910

Whole No. 193

CONTENTS

PAGE

The September Meeting of the San Francisco Section. By Professor C. A. NOBLE - - - - - - - 115

A New Proof of the Theorem of Weierstrass Concerning the Factorization of a Power Series. By Dr. W. D. MACMILLAN - - - - - - - - - - - 116

Kowalewski's Determinants. By Professor MAXIME BÔCHER 120

Differential Invariants . By Professor L. P. EISENHART - 140

Sturm's Geometrische Verwandtschaften. By Professor VIRGIL SNYDER - - - - - - - - - 150

Notes - - - - - - - - - - - - - 154

New Publications - - - - - - - - - 159

VOLUME XVII., NUMBER 4

JANUARY, 1911

Whole No. 194

CONTENTS

PAGE

The October Meeting of the American Mathematical Society. By Professor F. N. COLE - - - - - - - 167

The Königsberg Meeting of the Deutsche Mathematiker-Vereinigung - - - - - . - - - - 172

On the Saddlepoint in the Theory of Maxima and Minima and in the Calculus of Variations. By Professor R. G. D. RICHARDSON . - - - - - - - - - 177

Note on Identities Connecting Certain Integrals. By Dr. LOUIS INGOLD - - - - - - - - - 184

Poincaré's Göttingen Lectures. By Professor G. D. BIRKHOFF 190

The Theory of Electrons. By Professor E. B. WILSON . 194

Shorter Notices . - - - - - - - - - 200

Notes - - - - - - - - - - - - 212

New Publications - - - - - - - - - 216

VOLUME XVII., NUMBER 5

FEBRUARY, 1911

Whole No. 195

CONTENTS

PAGE

The Fourth Regular Meeting of the Southwestern Section. By Professor O. D. KELLOGG - - - - - 223

University Courses in Mathematics and the Master's Degree. Committee Report - - - - - - - - 230

Geometric Transformations. By Professor VIRGIL SNYDER 249

Shorter Notices - - - - - - - - - - 254

Notes - - - - - - - - - - - - 265

New Publications - - - - - - - - - 271

VOLUME XVII., NUMBER 6

MARCH, 1911

Whole No. 196

CONTENTS

PAGE

The Seventeenth Annual Meeting of the American Mathematical Society. By Professor F. N. COLE - - - 277

The Winter Meeting of the Chicago Section. By Professor H. E. SLAUGHT - - - - - - - - - 288

Preparation for Research and the Doctor's Degree in Mathematics. Committee Report - - - - - 305

Shorter Notices - - - - - - - - - - 312

Notes - - - - - - - - - - - - 323

New Publications - - - - - - - - - 326

VOLUME XVII., NUMBER 7

APRIL, 1911

Whole No. 197

CONTENTS

PAGE

Groups Generated by Two Operators Satisfying Two Conditions. By Professor G. A. MILLER - - - - - 333

Fundamental Regions for Cyclical Groups of Linear Fractional Transformations on Two Complex Variables. By Professor J. W. YOUNG - - - - - - - 340

On the Relative Discriminant of a Certain Kummer Field. By Professor JACOB WESTLUND - - - - - - 344

Note on Reciprocal Figures in Space. By Professor PETER FIELD - - - - - - - - - - - 347

Mathematical Physics for Engineers. By Professor E. B. WILSON - - - - - - - - - - 350

Shorter Notices - - - - - - - - - - 362

Notes - - - - - - - - - - - 378

New Publications - - - - - - - - - 384

VOLUME XVII., NUMBER 8

MAY, 1911

Whole No. 198

CONTENTS

PAGE

The February Meeting of the American Mathematical Society By Professor F. N. COLE - - - - - - - 389

On the Classification of Crystals. By Professor PAUL SAUREL 398

Horner's Method of Approximation Anticipated by Ruffini. By Professor FLORIAN CAJORI - - - - - - 409

The New Haven Colloquium Lectures. By Professor G. D. BIRKHOFF - - - - - - - - - - 414

Shorter Notices - - - - - - - - - - 428

Notes - - - - - - - - - - - 432

New Publications - - - - - - - - - 438

VOLUME XVII., NUMBER 9

JUNE, 1911

Whole No. 199

CONTENTS

PAGE

The April Meeting of the San Francisco Section. By Professor H. C. MORENO - - - - - - - - - 445
Invariant Conditions that a p-ary Form may have Multiple Linear Factors. By Professor O. E. GLENN - - 449
The General Term of a Recurring Series. By Professor ARTHUR RANUM - - - - - - - - - 457
Relations between the Gramian, the Wronskian, and a Third Determinant Connected with the Problem of Linear Dependence. By Professor D. R. CURTISS - - - 462
Note on Integration of Series by Lebesgue Integrals. By Mr. W. A. WILSON - - - - - - - - 467
Eisenhart's Differential Geometry. By Professor G. A. BLISS - - - - - - - - - - - 470
Stuyvaert's Studies in Analytical Geometry. By Dr. E. G. BILL - - - - - - - - - - - 479
Shorter Notices - - - - - - - - - - 484
Notes - - - - - - - - - - - - 490
New Publications - - - - - - - - - 496

VOLUME XVII., NUMBER 10

JULY, 1911

Whole No. 200

CONTENTS

PAGE

The Chicago Meeting of the American Mathematical Society. By Professor F. N. COLE - - - - - - - 505
On the Negative Discriminants for which there is a Single Class of Positive Primitive Binary Quadratic Forms. By Professor L. E. DICKSON - - - - - - - 534
Iterated Limits of Functions on an Abstract Range By Mr. R. E. ROOT - - - - - - - - - - 538
Note on a Mersenne Number. By Mr. H. J. WOODALL - 540
Shorter Notices - - - - - - - - - - 540
Correction - - - - - - - - - - - 550
Notes - - - - - - - - - - - - 550
New Publications - - - - - - - - - 555
Twentieth Annual List of Published Papers - - - 560
Index of Volume - - - - - - - - - 567

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

NOTE ON IMPLICIT FUNCTIONS DEFINED BY TWO EQUATIONS WHEN THE FUNCTIONAL DETERMINANT VANISHES.

BY PROFESSOR W. R. LONGLEY.

(Read before the American Mathematical Society, December 30, 1908.)

1. *Introduction.* Consider the two equations

$$(1)\quad f(x_1, x_2, \cdots, x_n; y, z) = 0, \quad g(x_1, x_2, \cdots, x_n; y, z) = 0,$$

and suppose a single point solution

$$(2)\qquad x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n; y = b, z = c,$$

is known. Under certain well-known conditions, of which one is the non-vanishing of the functional determinant $\partial(f, g)/\partial(y, z)$ at the point in question, we may affirm that equations (1) define uniquely the functions

$$(3)\qquad y = \phi(x_1, x_2, \cdots, x_n), \quad z = \psi(x_1, x_2, \cdots, x_n)$$

in the neighborhood of the system of values (2). In general if the functional determinant vanishes, the functions (3) are multiple valued.* There are, however, certain exceptional cases in which the determination of y and z as functions $x_1, x_2, \cdots, x_n$ is unique although the functional determinant vanishes. It is proposed in this paper to examine briefly some of these exceptional cases.

In a certain trivial way the corresponding exceptional cases exist also when we consider a single equation defining one dependent variable. Suppose an equation for the determination of y as a function of x has the form

$$(4)\qquad f(x, y) \equiv x f_1(x, y) = 0.$$

* The case of analytic functions was treated by Professor G. A. Bliss in his Princeton Colloquium Lectures (1909). These lectures have not yet been published.

Evidently $f = 0$ and $\partial f/\partial y = 0$ for $x = y = 0$, and the fundamental existence theorem is not applicable. But we may discard the factor x and if $f_1 = 0$, $\partial f_1/\partial y \neq 0$ for $x = y = 0$, we may affirm the existence of a unique solution of the form $y = \phi(x)$. Geometrically the locus defined by equation (4) in the neighborhood of the origin consists of two branches; but one of these is the curve $x = 0$, which is not expressible in the form $y = \phi(x)$. Again suppose the equation has the form

$$f(x, y) = f_1^2(x, y) = 0, \tag{5}$$

where $f_1(0, 0) = 0$. Here also $\partial f/\partial y = 0$ at the origin; but we may consider the equation $f_1 = 0$ and, if $\partial f_1/\partial y \neq 0$ for $x = y = 0$, we may affirm the existence of a unique solution of the form $y = \phi(x)$. Exceptional cases such as (4) and (5) for one equation are trivial because they are readily excluded by the usual assumption that f is irreducible in the arguments x and y.

In the case of two equations we shall assume always that each equation is irreducible; but this is not sufficient to exclude the exceptional cases which are similar, geometrically, to those presented by equations (4) and (5). The point of view is easily shown by some simple examples involving only one independent variable. Consider the two equations

$$f \equiv y - x = 0, \quad g \equiv 2yz - x^2 = 0. \tag{6}$$

Both equations are satisfied for $x = y = z = 0$ and $\partial(f, g)/\partial(y, z) = 0$ at the origin. From the first equation $y = x$ and, when this value of y is substituted in the second, we get $x(2z - x) = 0$. This equation is of the form (4) and has a unique solution $z = \frac{1}{2}x$. Hence for values of x different from zero the system (6) defines uniquely the functions

$$y = \phi(x) \equiv x, \quad z = \psi(x) \equiv \tfrac{1}{2}x. \tag{7}$$

Geometrically the curve defined by (6) consists of two branches passing through the origin; but one branch is the z-axis and is not expressible by equations of the form (7). Again consider the two equations

$$f \equiv z + 2y - 2x = 0, \; g \equiv 2z + 4y - 4x + 2yz - x^2 = 0. \tag{8}$$

Both equations are satisfied for $x = y = z = 0$, and $\partial(f, g)/\partial(y, z) = 0$ at the origin. From the first equation

$z = 2x - 2y$ and, when this value is substituted in the second, we get $(2y - x)^2 = 0$. This equation is of the form (5) and has a unique solution $y = \frac{1}{2}x$. Hence for values of x different from zero the system (8) defines uniquely the functions

$$(9) \qquad y = \phi(x) \equiv \tfrac{1}{2}x, \quad z = \psi(x) \equiv x.$$

Geometrically the curve defined by (8) consists of two coincident branches passing through the origin, that is, the two surfaces touch along the curve (9). Upon each surface the origin is an ordinary point.

The general method of investigating the solutions of equations (1), as illustrated by the preceding examples, is the following: Suppose one equation, say the first, can be solved by the fundamental theorem for one of the dependent variables, say y in terms of $z, x_1, \cdots, x_n$. When this value of y is substituted in the second equation it may happen that the latter becomes reducible in $z, x_1, \cdots, x_n$, and although the functional determinant vanishes the solution for z may be unique. For practical application it is convenient to formulate conditions upon f and g in order that this process shall lead to a unique result. Apparently no general formulation can be made, but it is possible to state theorems for special cases. Two of these theorems are given below.

2. In the theorem given in this section it is assumed that the functions f and g are real functions of real variables, and for convenience of reference the following statement of the fundamental theorem for two equations is given:

Fundamental Theorem. Consider the system of equations

$$(1) \quad f(x_1, x_2, \cdots, x_n; y, z) = 0, \quad g(x_1, x_2, \cdots, x_n; y, z) = 0,$$

and suppose a special solution

$$(2) \qquad x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n; \; y = b, z = c,$$

is known. Suppose the functions f and g are continuous, and possess first partial derivatives which are continuous, in the neighborhood of the system of values (2). Suppose the functional determinant

$$\Delta = \begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}$$

does not vanish for the system of values (2).

Then there exists one and only one system of continuous functions of the form

$$A: \quad y = \phi(x_1, x_2, \cdots, x_n), \quad z = \psi(x_1, x_2, \cdots, x_n)$$

which satisfy equations (1) and the condition

$$b = \phi(a_1, a_2, \cdots, a_n), \quad c = \psi(a_1, a_2, \cdots, a_n).$$

Moreover the implicit functions A thus defined possess first partial derivatives which are continuous in the neighborhood of the system of values $x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n$.

The solution described in the conclusion of this theorem will be referred to as a solution of the type A. When the functional determinant vanishes the following theorem states what seem to be the simplest conditions of practical value in order that there shall exist a unique solution of the type A. It is assumed in this theorem that f and g possess partial derivatives of the first and second* orders which are continuous in the neighborhood of the system of values (2).

THEOREM I. Consider equations (1) and suppose

$$(a) \qquad \begin{aligned} f(a_1, x_2, \cdots, x_n;\ b, z) &\equiv 0, \\ g(a_1, x_2, \cdots, x_n;\ b, z) &\equiv 0 \end{aligned} \qquad (\text{in } x_2, \cdots, x_n;\ z).$$

Suppose also

$$(b) \qquad \Delta_1 = \begin{vmatrix} f_y & f_{x_1} \\ g_y & g_{x_1} \end{vmatrix} = 0, \qquad (c) \quad \Delta_2 = \begin{vmatrix} 0 & f_y & f_{x_1} \\ f_y & f_{yx} & f_{xx_1} \\ g_y & g_{yx} & g_{xx_1} \end{vmatrix} \neq 0$$

for the system of values (2).

Then there exists one and only one solution of the type A.

To sketch the proof of this theorem we observe that condition (c) implies either $f_y \neq 0$ or $g_y \neq 0$. Assuming $f_y \neq 0$ we may apply the fundamental theorem for one equation to the first of equations (1) and obtain a solution for y. From the first of conditions (a) it follows that when $x_1 = a_1$, $y = b$ *identically* in $x_2, \cdots, x_n, z$. Hence the solution for y has the form

$$(10) \qquad y - b = (x_1 - a_1)h(x_1, x_2, \cdots, x_n;\ z).$$

When this value of y is substituted in the second of equations (1) it follows from (a) that the equation assumes the form

*It is not necessary to assume the existence and continuity of all the partial derivatives of the second order. No attempt is being made here to reduce these conditions to a minimum.

$$(11) \qquad (x_1 - a_1)\, g_1(x_1, x_2, \cdots, x_n;\ z) = 0.$$

Now the appropriate computation shows that [for the system of values (2)] condition (b) is equivalent to the condition $g_1 = 0$, and (c) is equivalent to the condition $\partial g_1/\partial z \neq 0$. Hence we may discard the factor $x_1 - a_1$ in equation (11) and apply the fundamental theorem for one equation to obtain a solution $z = \psi\,(x_1, x_2, \cdots, x_n)$. When this value of z is substituted in (10) we have the final solution in the form

$$(12) \qquad y = \phi\,(x_1, x_2, \cdots, x_n), \quad z = \psi\,(x_1, x_2, \cdots, x_n).$$

Each step in the process yields a unique result; hence the solution (12) is unique and of type A.

3. An example* of a system of equations of the type to which Theorem I is applicable may be found in a problem in the calculus of variations. The problem has been treated† in a paper by G. A. Bliss in the BULLETIN, volume 13 (1907), page 321, and the system referred to is (3), page 322. The equations to be solved for s and t are

$$(13) \qquad f \equiv \phi(s, t) - r\cos\alpha = 0, \quad g \equiv \psi(s, t) - r\sin\alpha = 0,$$

where the functions ϕ and ψ satisfy the conditions

$$(14) \qquad \begin{aligned} &\phi(0, t) \equiv 0, \qquad \psi(0, t) \equiv 0, \\ &\phi_s(0, t) \equiv \cos t, \quad \psi_s(0, t) \equiv \sin t. \end{aligned}$$

Evidently a particular solution of equations (13) is

$$(15) \qquad r = 0, \quad \alpha = k, \quad s = 0, \quad t = k,$$

where k is any constant.

For the system of values (15) the functional determinant

$$\Delta = \begin{vmatrix} \phi_s & \phi_t \\ \psi_s & \psi_t \end{vmatrix} = \begin{vmatrix} \cos k & 0 \\ \sin k & 0 \end{vmatrix} = 0,$$

and the fundamental theorem is not applicable. We apply then the hypothesis of Theorem I. The identities (14) show

*Other examples are to be found in papers by F. R. Moulton, "A class of periodic solutions, etc.," *Transactions*, vol. 7 (1906), p. 546, equations (16); and by W. R. Longley, "A class of periodic orbits, etc.," *Transactions*, vol. 8 (1907), p. 166, equations (15).

†See also Bolza, Variationsrechnung, p. 270.

that condition (a) is satisfied. Also

$$\Delta_1 = \begin{vmatrix} f_x & f_r \\ g_x & g_r \end{vmatrix} = \begin{vmatrix} \cos k & -\cos k \\ \sin k & -\sin k \end{vmatrix} = 0,$$

$$\Delta_2 = \begin{vmatrix} 0 & f_x & f_r \\ f_x & f_{xx} & f_{xr} \\ g_x & g_{xx} & g_{xr} \end{vmatrix} = \begin{vmatrix} 0 & \cos k & -\cos k \\ \cos k & -\sin k & 0 \\ \sin k & \cos k & 0 \end{vmatrix} = -\cos k.$$

Hence if $\cos k \neq 0$ the hypothesis of Theorem I is satisfied and we may affirm the existence of a unique solution of the type A. If $\cos k = 0$, then $\sin k \neq 0$ and the apparent difficulty is overcome by interchanging the notation f and g in equations (13).

4. If the functions f and g in Theorem I are analytic, the process of solving the first of equations (1) for y, substituting in the second, and solving for z shows that the solution for y and z is also analytic. The solution may be obtained, however, without going through this process. For simplicity this will be shown when there is only one independent variable x, and we will suppose that the equations are satisfied for $x = y = z = 0$. If f and g are regular at the origin, equations (1) may be written

$$(16) \qquad f = \Sigma a_{ijk} x^i y^j z^k = 0, \quad g = \Sigma b_{ijk} x^i y^j z^k = 0.$$

For the system (16) the conditions (a), (b), and (c) are

$$(a) \qquad a_{00k} = 0, \quad b_{00k} = 0 \qquad (k = 1, 2, \cdots),$$

$$(b)\ \Delta_1 = \begin{vmatrix} a_{100} & a_{010} \\ b_{100} & b_{010} \end{vmatrix} = 0, \qquad (c)\ \Delta_2 = \begin{vmatrix} 0 & a_{010} & a_{100} \\ a_{010} & a_{011} & a_{101} \\ b_{010} & b_{011} & b_{101} \end{vmatrix} \neq 0.$$

The assumption (c) implies that either $a_{010} \neq 0$ or $b_{010} \neq 0$. Supposing a_{010} is not zero, we may without loss of generality take it equal to unity (by multiplying the series for f by $1/a_{010}$) and we may also take $\Delta_2 = 1$ (by multiplying the series for g by $1/\Delta_2$).

The solution to be determined has the form

$$(17) \qquad y = \alpha_1 x + \alpha_2 x^2 + \cdots, \quad z = \beta_1 x + \beta_2 x^2 + \cdots.$$

When the values of y and z from (17) are substituted in (16) the coefficient of each power of x must vanish. Equating to

zero the coefficients of the first power of x we have

$$a_{100} + \alpha_1 = 0, \quad b_{100} + b_{010}\alpha_1 = 0.$$

From (b) it follows that these two equations are consistent and hence $\alpha_1 = -a_{100}$.

Equating to zero the coefficients of x^2, we have

$$\begin{aligned}(a_{101} + a_{011}\alpha_1)\beta_1 + \quad \alpha_2 + a_{200} + a_{020}\alpha_1^2 + a_{110}\alpha_1 = 0,\\ (b_{101} + b_{011}\alpha_1)\beta_1 + b_{010}\alpha_2 + b_{200} + b_{020}\alpha_1^2 + b_{110}\alpha_1 = 0.\end{aligned} \tag{18}$$

When the value of α_1 is substituted in (18) it is found that the determinant of the coefficients of α_2 and β_1 is equal to $\Delta_2(=1)$. Hence equations (18) determine α_2 and β_1 uniquely as rational integral functions of the coefficients of (16) of order * less than 3.

Proceeding in this way, we may determine step by step the coefficients in the series (17). For suppose

$$\alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-2}$$

have been determined as rational integral functions of the coefficients of (16) of order less than n. Equating to zero the coefficients of x^n we have

$$(a_{101} - a_{011}a_{100})\beta_{n-1} + \alpha_n + \phi_n = 0,$$

$$(b_{101} - b_{011}a_{100})\beta_{n-1} + b_{010}\alpha_n + \psi_n = 0,$$

where ϕ_n, ψ_n denote rational integral functions of a_{ijk}, b_{ijk} of order less than $n+1$. The determinant of the coefficients of β_{n-1} and α_n is equal to unity and hence these quantities are determined uniquely as rational integral functions of the coefficients of (16) of order less than $n+1$.

5. A method of extending the conditions of Theorem I is suggested by considering the geometric interpretation. Suppose the equations involve three variables x, y, z, which will be interpreted as rectangular coordinates in space. The two surfaces $f = 0$, $g = 0$ then define a curve and we suppose the hypothesis of Theorem I is satisfied at a point which will be taken for the origin. Then from the conditions of the theorem it follows that the curve consists of two branches passing through the origin; but one of these branches is the z-axis and is not expressible by equations of the form $y = \phi(x)$, $z = \psi(x)$.

*By the order of a coefficient is meant the number $i+j+k$.

It is apparent geometrically that if the curve consists of n branches passing through the origin, and if $n-1$ of these branches are plane curves lying in the yz-plane, while one branch does not lie in this plane, then the equations $f=0$, $g=0$ will have one and only one solution of the form $y=\phi(x)$, $z=\psi(x)$. When these geometric conditions are stated in analytic language we may formulate further theorems similar to Theorem I. Theorem II is given as an example.

THEOREM II. Consider equations (1) and suppose

$$(\alpha) \qquad \begin{aligned} &f(a_1, x_2, \cdots, x_n;\ \lambda(z), z) \equiv 0, \\ &g(a_1, x_2, \cdots, x_n;\ \lambda(z), z) \equiv 0, \end{aligned} \qquad (\text{in } x_2, \cdots, x_n;\ z)$$

where the function $\lambda(z)$, which is substituted for y, is continuous and has a continuous first derivative in the neighborhood of $z=c$, and $b=\lambda(c)$. Suppose also

$$(\beta) \qquad \Delta_1 = \begin{vmatrix} f_y & f_{x_1} \\ g_y & g_{x_1} \end{vmatrix} = 0,$$

$$(\gamma) \qquad \Delta_2 = \begin{vmatrix} 0 & f_y & f_{x_1} \\ f_y & f_{yz} + \lambda' f_{yy} & f_{zx_1} + \lambda' f_{yx_1} \\ g_y & g_{yz} + \lambda' g_{yy} & g_{zx_1} + \lambda' g_{yx_1} \end{vmatrix} \neq 0,$$

for the system of values (2). (The derivative of λ is denoted by λ'.)

Then there exists one and only one solution of the type A.

In order to prove this theorem we set

$$(19) \qquad y = \eta + \lambda(z)$$

in equations (1) and obtain the set

$$F(x_1, x_2, \cdots, x_n;\ \eta, z) \equiv f(x_1, x_2, \cdots, x_n;\ \eta + \lambda(z), z) = 0,$$
$$G(x_1, x_2, \cdots, x_n;\ \eta, z) \equiv g(x_1, x_2, \cdots, x_n;\ \eta + \lambda(z), z) = 0.$$

The condition (α) becomes

$$(a) \qquad \begin{aligned} &F(a_1, x_2, \cdots, x_n;\ 0, z) \equiv 0, \\ &G(a_1, x_2, \cdots, x_n;\ 0, z) \equiv 0, \end{aligned} \qquad (\text{in } x_2, \cdots, x_n;\ z)$$

and the conditions (β) and (γ) become, respectively,

$$(b) \quad \begin{vmatrix} F_\eta & F_{z_1} \\ G_\eta & G_{z_1} \end{vmatrix} = 0, \qquad (c) \quad \begin{vmatrix} 0 & F_\eta & F_{z_1} \\ F_\eta & F_{\eta\eta} & F_{\eta z_1} \\ G_\eta & G_{\eta z} & G_{z z_1} \end{vmatrix} \neq 0$$

for the system of values

$$x_1 = a_1, \quad x_2 = a_2, \cdots, x_n = a_n; \ \eta = 0, \quad z = c.$$

Hence the hypothesis of Theorem I is satisfied for the equations $F = 0$, $G = 0$, and we have a unique solution of the form

$$(20) \qquad \eta = \eta(x_1, x_2, \cdots, x_n), \quad z = \psi(x_1, x_2, \cdots, x_n),$$

such that

$$0 = \eta(a_1, a_2, \cdots, a_n), \quad c = \psi(a_1, a_2, \cdots, a_n).$$

Substituting the values of η and z in (19), we have the required solution for y.

If there is only one independent variable x, the conditions of Theorem II, for the system of values $x = y = z = 0$, imply that the equations have the form

$$f(x, y, z) \equiv [y - \lambda(z)] f_1(x, y, z) + x f_2(x, y, z) = 0,$$

$$g(x, y, z) \equiv [y - \lambda(z)] g_1(x, y, z) + x g_2(x, y, z) = 0.$$

The curve defined by these equations consists of two branches passing through the origin. One branch is the curve $y = \lambda(z)$ in the yz-plane and is not expressible by equations of the form $y = \phi(x)$, $z = \psi(x)$.

SHEFFIELD SCIENTIFIC SCHOOL,
May, 1910.

STURM'S METHOD OF INTEGRATING $dx/\sqrt{X} + dy/\sqrt{Y} = 0$.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society, April 30, 1910.)

ONE of the simplest methods of obtaining the addition theorem for elliptic integrals of the first kind is based upon a method of integration which is usually referred to as Sturm's

method, and was thus communicated by Liouville in the *Comptes rendus*, 1856, No. 21, page 988. But on page 1087 of the same volume Liouville stated that the method was due to Despeyrous, and in the Cours d'analyse of Ch. Sturm, Paris, 1877, volume 2, page 340, the method is given in a note "par M. Despeyrous." The object of the present paper is to discuss certain details of Sturm's method and the extent to which it may be employed, without considering the addition theorems. The method is applicable to special types of the equation

$$dx/\sqrt{f(x)} + dy/\sqrt{f(y)} = 0, \tag{1}$$

and since the variables in (1) are separated it can be integrated first term by term. By Sturm's method a second integral of (1) is sometimes obtainable by clearing of fractions and in most cases using also an integrating factor. Integration by parts is performed on each term of the new form of (1) and the two indicated integrations at this stage are to be combined under one integral sign; then if possible by the use of (1) this integrand is to be made to vanish. When this method is successful a comparison of the resulting integral with that first obtained leads to an addition theorem. Illustrations may be found in Byerly, Integral Calculus, page 235, and Durège, Elliptische Funktionen, 4th edition, page 101.

So far as the writer has discovered, in all cases where this method has been used the form of $f(x)$ has been considered known, and from the types x^2, $1-x^2$, $(1+x^2)^2$, $(1-x^2)(1-k^2x^2)$ follow addition theorems for $\log x$, $\sin^{-1}x$, $\tan^{-1}x$, and $\mathrm{sn}^{-1}x$ respectively. The last two types require an additional integrating factor but the first two do not.

The following is from Schlömilch, Compendium der höheren Analysis, volume 2, page 326: "Es sei * * *

$$dx/\sqrt{(1-x^2)} + dy/\sqrt{(1-y^2)} = 0 \tag{2}$$

oder

$$\sqrt{(1-y^2)}dx + \sqrt{(1-x^2)}dy = 0, \tag{3}$$

mithin

$$\int \sqrt{(1-y^2)}dx + \int \sqrt{(1-x^2)}dy = \text{const.} \tag{4}$$

Bei theilweiser Integration ist ferner

$$(5)\qquad \begin{aligned} \int \sqrt{(1-y^2)}dx &= \sqrt{(1-y^2)}\,x + \int \frac{xydy}{\sqrt{(1-y^2)}}, \\ \int \sqrt{(1-x^2)}dy &= \sqrt{(1-x^2)}\,y + \int \frac{yxdx}{\sqrt{(1-x^2)}} \end{aligned}$$

mithin durch Addition

$$(6)\quad x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}+\int xy\left[\frac{dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}\right]=\text{const.}$$

Wegen Nro. (2) verschwindet das Integral und es bleibt einfacher

$$(7)\qquad x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}=c, \quad * \quad * \quad *."$$

Though (7) is an integral of (2) it should be noticed that (3) is not an exact differential, and the integral in (6) for the analogous case $f(x) = x^2$ is a constant and does not vanish.

Schröter in the *Zeitschrift für Mathematik und Physik*, volume 17 (1872), page 508, avoids the difficulty by an inverse method which, applied to the example above, gives

$$(8)\qquad \begin{aligned} &d[x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}] \\ &=[\sqrt{(1-y^2)}dx+\sqrt{(1-x^2)}dy]-xy\left[\frac{dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}\right] \\ &=\left[\frac{dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}\right][\sqrt{(1-x^2)}\sqrt{(1-y^2)}-xy]. \end{aligned}$$

Thus from (2) equation (8) becomes

$$(9)\qquad d[x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}]=0,$$

whence the integral (7) follows. In this case no additional integrating factor is necessary.

The use of Sturm's method in integrating

$$(10)\qquad \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}+\frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}=0$$

has been called "elegant" by several writers who, however, have felt that the form of the integrating factor, after clearing of fractions, i. e., $1/(1-k^2x^2y^2)$, is not obvious from the form of equation (10).

Schröter makes changes of variables in (10) and then shows how to determine the factor. But in the present paper the factor will be expressed, under certain sufficient conditions, as a

function of $f(x)$ and $f(y)$ in the general type, equation (1), and then the corresponding form of $f(x)$ will be determined.

From (1), after supplying the unknown factor $F(x, y)$,

$$\frac{\sqrt{f(y)}dx}{F} + \frac{\sqrt{f(x)}dy}{F} = 0. \tag{11}$$

The use of Sturm's method in its modified form gives

$$\begin{aligned} &d\left[\frac{x\sqrt{f(y)}}{F} + \frac{y\sqrt{f(x)}}{F}\right] \\ &= [x\sqrt{f(y)} + y\sqrt{f(x)}]\left[\frac{\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy}{-F^2}\right] \\ &+ dx\left[\frac{\sqrt{f(y)} + \frac{yf'(x)}{2\sqrt{f(x)}}}{F}\right] + dy\left[\frac{\sqrt{f(x)} + \frac{xf'(y)}{2\sqrt{f(y)}}}{F}\right] \\ &= \frac{dx}{\sqrt{f(x)}}\left[-\frac{x\sqrt{f(x)}\sqrt{f(y)}}{F^2}\frac{\partial F}{\partial x} - \frac{yf(x)}{F^2}\frac{\partial F}{\partial x}\right. \\ &\qquad\qquad \left. + \frac{\sqrt{f(x)}\sqrt{f(y)}}{F} + \frac{yf'(x)}{2F}\right] \\ &+ \frac{dy}{\sqrt{f(y)}}\left[-\frac{y\sqrt{f(x)}\sqrt{f(y)}}{F^2}\frac{\partial F}{\partial y} - \frac{xf(y)}{F^2}\frac{\partial F}{\partial y}\right. \\ &\qquad\qquad \left. + \frac{\sqrt{f(x)}\sqrt{f(y)}}{F} + \frac{xf'(y)}{2F}\right]. \end{aligned} \tag{12}$$

The last member of (12) will vanish in virtue of (1), provided the coefficients of $dx/\sqrt{f(x)}$ and $dy/\sqrt{f(y)}$ are equal, and the sufficient conditions are these:

$$x\frac{\partial F}{\partial x} = y\frac{\partial F}{\partial y}. \tag{13}$$

$$yf(x)\frac{\partial F}{\partial x} - \frac{yFf'(x)}{2} = xf(y)\frac{\partial F}{\partial y} - \frac{xFf'(y)}{2}. \tag{14}$$

From (13) it follows that F is a function of the product xy, which is an important result and permits the reduction of (14) to

$$2x^2F'f(y) - xFf'(y) = 2y^2F'f(x) - yFf'(x), \tag{15}$$

whence

$$\frac{2F'}{F} = \frac{xf'(y) - yf'(x)}{x^2 f(y) - y^2 f(x)}. \tag{16}$$

Since F is a function of xy, the second member of (16) must also be a function of xy, and for proper forms of $f(x)$, F may be found from (16) by integration. An illustration is $f(x) = (1 - x^2)(1 - k^2x^2)$, then F is $c(1 - k^2x^2y^2)$.

The determination of the form of $f(x)$ is the next important step. Let the second member of (16) be called V; then employing the condition that V shall be a function of xy, i. e.,

$$x\frac{\partial V}{\partial x} = y\frac{\partial V}{\partial y}, \tag{13a}$$

the result after some reduction is

$$\begin{aligned}&\frac{f(x)f'(x)}{x^3} - \frac{f(y)f'(y)}{y^3} + \frac{f(x)f''(x)}{x^2} - \frac{f(y)f''(y)}{y^3}\\ &+ \frac{f'^2(y)}{y^2} - \frac{f'^2(x)}{x^2} + \frac{3f'(x)f(y)}{xy^2} - \frac{3f'(y)f(x)}{yx^2}\\ &+ \frac{f(x)f''(y)}{x^2} - \frac{f(y)f''(x)}{y^2} = 0.\end{aligned} \tag{17}$$

When (17) is differentiated partially with respect to x and y in succession, the first six terms vanish and the remaining four give after reduction

$$\begin{aligned}&\left[\frac{f'(x)}{x^2} - \frac{2f(x)}{x^3}\right]\left[f'''(y) - \frac{3f''(y)}{x} + \frac{3f'(y)}{y^2}\right]\\ &= \left[\frac{f'(y)}{y^2} - \frac{2f(y)}{y^3}\right]\left[f'''(x) - \frac{3f''(x)}{x} + \frac{3f'(x)}{x^2}\right].\end{aligned} \tag{18}$$

The variables in (18) may now be separated by division, and since the result is that a function of x equals a function of y, each function is a constant, i. e.,

$$\frac{f'''(x) - \dfrac{3f''(x)}{x} + \dfrac{3f'(x)}{x^2}}{\dfrac{f'(x)}{x^2} - \dfrac{2f(x)}{x^3}} = 4(a^2 - 1). \tag{19}$$

The form of the constant is chosen so that later results will be more simple, and the general solution of (19) is

$$f(x) = \alpha x^{2(1-a)} + \beta x^2 + \gamma x^{2(1+a)}. \tag{20}$$

From (16) and (20),

$$F = C_1(\alpha x^{1-a}y^{1-a} - \gamma x^{1+a}y^{1+a}). \tag{21}$$

Having satisfied the conditions expressed by (13) and (14), the last member of (12) vanishes and the first member becomes an exact differential, whence integration and substitution of f and F from (20) and (21) give

$$\frac{x\sqrt{(\alpha y^{2(1-a)} + \beta y^2 + \gamma y^{2(1+a)})} + y\sqrt{(\alpha x^{2(1-a)} + \beta x^2 + \gamma x^{2(1+a)})}}{\alpha x^{1-a}y^{1-a} - \gamma x^{1+a}y^{1+a}} = C, \tag{22}$$

which is an integral of

$$\frac{dx}{\sqrt{(\alpha x^{2(1-a)} + \beta x^2 + \gamma x^{2(1+a)})}} + \frac{dy}{\sqrt{(\alpha y^{2(1-a)} + \beta y^2 + \gamma y^{2(1+a)})}} = 0. \tag{23}$$

Making $a = 1$, or replacing x^a by x, and y^a by y, which may be shown to have the same effect, gives what may be considered the standard forms

$$\frac{x\sqrt{(\alpha + \beta y^2 + \gamma y^4)} + y\sqrt{(\alpha + \beta x^2 + \gamma x^4)}}{\alpha - \gamma x^2 y^2} = C, \tag{24}$$

$$\frac{dx}{\sqrt{(\alpha + \beta x^2 + \gamma x^4)}} + \frac{dy}{\sqrt{(\alpha + \beta y^2 + \gamma y^4)}} = 0. \tag{25}$$

The denominator in (24) is the integrating factor sought, and is a simple function of α and γ in (25), being independent of β. By giving proper values to α, β, and γ in (24) and (25) addition formulas may be obtained for the logarithm, inverse sine and tangent (both circular and hyperbolic), and elliptic integrals of the first kind. An illustration of the use of the apparently more general forms (22) and (23) is obtained by taking $a = \frac{1}{2}$, then $f(x)$ is $\alpha x + \beta x^2 + \gamma x^3$, which includes $x(1 - x)(1 - \kappa x)$, favored by Klein as a canonical form in elliptic integrals.

In Schröter's article (loc. cit.) the starting point is

$$d\phi/\Delta\phi + d\psi/\Delta\psi = 0 \qquad [\Delta\phi = \sqrt{(1 - k^2 \sin^2\phi)}]. \tag{26}$$

Thus the form of $f(\phi)$ is assigned in advance, and (26) is inte-

grated by Sturm's method as three distinct problems, each being obtained by a change of variables. Although the Legendrian notation is practically adhered to, the respective substitutions are equivalent to

$$(27) \qquad 1 - k^2 \sin^2 \phi = x^2, \quad \cos^2 \phi = x^2, \quad \sin^2 \phi = x^2.$$

The respective values of $f(x)$ are equivalent to

$$(28) \quad (1 - x^2)(x^2 - k'^2), \ (1 - x^2)(k'^2 + k^2 x^2), \ (1 - x^2)(1 - k^2 x^2).$$

Schröter's values of F are deduced for each of the three cases independently, by expressing the sufficient conditions for the vanishing of coefficients in his equation corresponding to (12) above. His results are obtainable at once by using his respective values of $f(x)$ from (28) in (25) and (24), and this affords a concrete illustration of the theorem which these two equations express.

UNIVERSITY OF PENNSYLVANIA,
April, 1910.

A PROPERTY OF A SPECIAL LINEAR SUBSTITUTION.

BY PROFESSOR F. R. SHARPE.

LET ξ_i denote homogeneous line coordinates * which satisfy the quadratic identity

$$(1) \qquad \Sigma \xi_i^2 = 0.$$

The condition that two lines ξ, ξ' intersect is

$$(2) \qquad \Sigma \xi_i \xi'_i = 0.$$

The equation of a linear complex is of the form

$$(3) \qquad \Sigma a_i \xi_i = 0.$$

It is clear from the above equations that, when

$$(4) \qquad \Sigma a_i^2 = 0,$$

the lines of (3) all intersect the line a; (3) is then called a special complex.

* Jessop: Treatise on the line complex, Arts 9, 15–20.

If

$$\lambda = -\frac{2\Sigma a_i x_i}{\Sigma a_i^2},$$

then

$$\Sigma(x_i + \lambda a_i)^2 = \Sigma x_i^2.$$

Hence

(5) $$\Sigma x_i' \xi_i = \Sigma(2\Sigma a_i x_i \cdot a_i - \Sigma a_i^2 \cdot x_i)\xi_i = 0$$

is a special complex if

(6) $$\Sigma x_i \xi_i = 0$$

is a special complex.

The linear complex (5) may be termed the transform of (6) through (3).

Two complexes x and y are said to be in involution if

(7) $$\Sigma x_i y_i = 0.$$

When the two complexes are transformed through a

(8) $$\Sigma x' y' = (\Sigma a_i^2)^2 \cdot \Sigma x_i y_i = 0.$$

Hence the transforms are also in involution. Consider the transformations

(A) $$x_i' = 2\Sigma a_i x_i \cdot a_i - \Sigma a_i^2 \cdot x_i,$$

(B) $$x_i'' = 2\Sigma b_i x_i' \cdot b_i - \Sigma b_i^2 \cdot x_i',$$

which transform x through the complexes a and b respectively, and

(A') $$x_i''' = 2\Sigma(sa_i + tb_i)x_i \cdot (sa_i + tb_i) - \Sigma(sa_i + tb_i)^2 \cdot x_i,$$

(B') $$x_i^{\text{IV}} = 2\Sigma(s'a_i + t'b_i)x_i''' \cdot (s'a_i + t'b_i) - \Sigma(s'a_i + t'b_i)^2 \cdot x_i''',$$

which transform x through two complexes of the pencil defined by a and b. These transformations have the property that if A, B, and A' are given, then B' can be uniquely determined so that either

(9) $$AB = A'B'$$

or

(9)* $$AB = B'A'.$$

This can be proved very simply as follows. The coefficient of x_j in AB is

$$4a_j b_i \Sigma a_i b_i - 2a_j a_i \Sigma b_i^2 - 2b_j b_i \Sigma a_i^2.$$

If (9) is satisfied, the corresponding coefficients must have a common ratio ρ, so that

$$x_i'' \equiv \rho x_i^{IV}. \tag{10}$$

Hence

$$\rho = \frac{\text{coefficient of } x_j \text{ in } x_i''}{\text{coefficient of } x_j \text{ in } x_i^{IV}} = \frac{\text{coefficient of } x_i \text{ in } x_j''}{\text{coefficient of } x_i \text{ in } x_j^{IV}}.$$

Subtracting the numerators and denominators, we find after a little reduction

$$\rho = \frac{\Sigma a_i b_i}{(st' - s't)\Sigma(sa_i + tb_i, s'a_i + t'b_i)}. \tag{11}$$

Similarly by addition we find

$$\rho = \frac{1}{(st' - s't)^2}. \tag{12}$$

From (11) and (12) it follows that

$$\Sigma(sa_i + tb_i, s'a_i + t'b_i) = (st' - s't)\Sigma a_i b_i, \tag{13}$$

that is

$$ss'\Sigma a_i^2 + 2s't\Sigma a_i b_i + tt'\Sigma b_i^2 = 0. \tag{13}^*$$

Solving, we have

$$\frac{s'}{t'} = \frac{-t\Sigma b_i^2}{s\Sigma a_i^2 + 2t\Sigma a_i b_i}. \tag{14}$$

If we take (9)* instead of (9), then (13) becomes

$$\Sigma(sa_i + tb_i, s'a' + t'b_i) = (s't - st')\Sigma a_i b_i \tag{15}$$

and (14) becomes

$$\frac{s'}{t'} = \frac{t\Sigma b_i^2 + 2s\Sigma a_i b_i}{-s\Sigma a_i^2}. \tag{16}$$

The equations (13) and (15) are both included in

$$\{\Sigma(sa_i + tb_i, s'a_i + t'b_i)\}^2 = (st' - s't)^2\{\Sigma^2 a_i b_i\}^2, \tag{17}$$

but

$$\{\Sigma(sa_i + tb_i, s'a_i + t'b_i)\}^2 - \Sigma(sa_i + tb_i)^2 \cdot \{\Sigma(s'a_i + t'b_i)^2 = (st' - s't)^2\{(\Sigma a_i b_i)^2 - \Sigma a_i^2 \cdot \Sigma b_i^2\}. \tag{18}$$

Hence (17) may be written in the form

$$\frac{\{\Sigma(sa_i + tb_i,\ s'a_i + t'b_i)\}^2}{\Sigma(sa_i + tb_i)^2 \cdot \Sigma(s'a_i + t'b_i)^2} = \frac{\{\Sigma a_i b_i\}^2}{\Sigma a_i^2 \cdot \Sigma b_i^2}. \tag{19}$$

It is important to notice that the roots of the quadratic (19) in s'/t' are rationally separable and are given by (14) and (16), which are the solutions of (9) and (9)* respectively.

It remains to show that

$$\rho = \frac{\text{coefficient of } x_i \text{ in } x_i''}{\text{coefficient of } x_i \text{ in } x_i^{\text{IV}}}. \tag{20}$$

The coefficients are similar to those which were used in (11) and (12), but have an extra term

$$\Sigma a_i^2 \cdot \Sigma b_i^2 \quad \text{and} \quad \Sigma(sa_i + tb_i)^2 \cdot \Sigma(s'a_i + t'b_i)^2 \tag{21}$$

respectively. From (17) and (18) it follows that the terms (21) are also in the ratio 1 to $(st' - s't)^2$. This completes the proof of the property. The property may be expressed in terms of the geometry of the sphere.*

If we regard the x_i as point coordinates in hyperspace, then (5) shows that the quadratic variety

$$\Sigma x_i^2 = 0$$

is invariant under the transformation and (8) shows that polarity is preserved. If in (13)* we put

$$s = 0, \quad t' = 0,$$

we find that the condition that A and B are commutative is

$$\Sigma a_i b_i = 0.$$

Hence it follows that A and B are commutative when a and b are in involution.

CORNELL UNIVERSITY,
March, 1910.

* P. F. Smith, *Transactions*, vol. 1 (1900), p. 378, footnote. The statement in the footnote overlooks the distinction between our equations (14), (16).

ON THE FACTORIZATION OF INTEGRAL FUNCTIONS WITH p-ADIC COEFFICIENTS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, September 6, 1910.)

1. If $F(x)$ is an integral function of degree n with integral p-adic coefficients, then for any integer k we have a congruence

$$F(x) \equiv F^{(k)}(x) = F_0(x) + pF_1(x) + p^2F_2(x) + \cdots + p^kF_k(x) \pmod{p^{k+1}}, \tag{1}$$

in which each $F_i(x)$ is an integral function of degree $\leqq n$ with coefficients belonging to the set $0, 1, \cdots, p-1$. The function $F^{(k)}(x)$ is called the convergent of rank k of $F(x)$. If

$$F(x) = f(x) \cdot g(x) \quad (p), \tag{2}$$

in which the factors are integral functions with integral p-adic coefficients, then for any integer k we obviously have

$$F^{(k)}(x) \equiv f^{(k)}(x) \cdot g^{(k)}(x) \pmod{p^{k+1}}. \tag{3}$$

The following converse theorem plays a fundamental rôle in Hensel's new theory of algebraic numbers:* *If*

$$F(x) \equiv f_0(x) \cdot g_0(x) \pmod{p^{s+1}} \tag{4}$$

for † $s + 1 > 2\rho$, *where* ρ *is the order of the resultant* R *of* $f_0(x)$ *and* $g_0(x)$, *then* $F(x)$ *is the product* (2) *of two integral functions with integral p-adic coefficients whose convergents of rank* $s - \rho$ *are* $f_0(x)$ *and* $g_0(x)$.

Hensel's proof is in effect a process to construct the successive convergents of $f(x)$ and $g(x)$. Each step of the process requires the solution of a linear equation in two unknowns with p-adic coefficients. The object of this note is to furnish a decidedly simpler process, which dispenses with these linear

* Hensel, Theorie der algebraischen Zahlen, Leipzig, Teubner, 1908, p. 71.

† This condition is satiafied if $s = \delta$, where δ is the order of the discriminant of $F(x)$. Hence we obtain as a corollary the theorem of Hensel, page 68.

equations, and requires only the solution of a single linear congruence.

By the remark of Hensel, bottom of page 64, we may assume that the leading coefficients in $F(x)$, $f_0(x)$, and $g_0(x)$ are powers of p, so that $F - f_0 g_0$ is of degree less than n.

2. For the sake of simplicity, we first establish the theorem for the important case $s = \rho = 0$: *If* $F(x) = F_0(x) + pF_1(x) + \cdots$, *in which the coefficient of* x^n *in* F_0 *is unity and* $F_i (i > 0)$ *is of degree less than* n, *and if*

$$F_0(x) \equiv f_0(x) \cdot g_0(x) \pmod{p}, \tag{5}$$

in which f_0 *and* g_0 *are integral functions of degrees* μ *and* ν *respectively, with integral coefficients, while* f_0 *and* g_0 *have no common factor modulo* p, *then integral functions* $f_i(x)$ *of degrees* $< \mu$ *and* $g_i(x)$ *of degrees* $< \nu$ *with integral coefficients can be so determined that*

$$f = f_0 + pf_1 + p^2 f_2 + \cdots, \quad g = g_0 + pg_1 + p^2 g_2 + \cdots.$$

have as their product $F(x)$. It is meant by the last statement that congruence (3) holds for every integer k.

Since f_0 and g_0 have no common factor, integral functions $a(x)$ and $b(x)$ with integral coefficients can be determined (for example, by Euclid's process) such that

$$af_0 + bg_0 \equiv 1 \pmod{p}. \tag{6}$$

By (5), $F_0 - f_0 g_0$ is the product of p by an integral function with integral coefficients which may be designated $L_1(x) - F_1(x)$,

$$F_0 - f_0 g_0 = p(L_1 - F_1). \tag{7}$$

By the remark at the end of § 1, L_1 like F_1 is of degree less than n. The congruence

$$F_0 + pF_1 \equiv (f_0 + pf_1)(g_0 + pg_1) \pmod{p^2}$$

is equivalent, in view of (7), to

$$L_1 \equiv f_0 g_1 + g_0 f_1 \pmod{p}. \tag{8}$$

In view of (6), every set of solutions is given by

$$f_1 \equiv bL_1 - \rho_1 f_0, \quad g_1 \equiv aL_1 + \rho_1 g_0. \tag{9}$$

We choose $\rho_1(x)$ so that the degree of f_1 shall be less than the degree μ of f_0. Then the final term of (8) is of degree $< \mu + \nu$,

so that the degree of g_1 is $< \nu$. Hence the required functions f_1 and g_1 are given by (9).

To make the general step by induction, suppose that, for $i = 1, \cdots, k$, functions f_i of degrees $< \mu$ and g_i of degrees $< \nu$ have been determined so that (3) holds. Hence we may set

$$F^{(k)} = f^{(k)}g^{(k)} + p^{k+1}(L_{k+1} - F_{k+1}), \tag{10}$$

where L_{k+1} is of degree less than n. Since

$$\begin{gathered} F^{(k+1)} = F^{(k)} + p^{k+1}F_{k+1}, \quad f^{(k+1)} = f^{(k)} + p^{k+1}f_{k+1}, \\ g^{(k+1)} = g^{(k)} + p^{k+1}g_{k+1}, \end{gathered} \tag{11}$$

the condition for the congruence

$$F^{(k+1)} \equiv f^{(k+1)}g^{(k+1)} \pmod{p^{k+2}}$$

becomes, upon applying (10),

$$L_{k+1} \equiv f^{(k)}g_{k+1} + g^{(k)}f_{k+1} \equiv f_0g_{k+1} + g_0f_{k+1} \pmod{p}.$$

The general set of solutions is

$$f_{k+1} \equiv bL_{k+1} - \rho_{k+1}f_0, \quad g_{k+1} \equiv aL_{k+1} + \rho_{k+1}g_0.$$

We determine ρ_{k+1} so that the degree of f_{k+1} shall be less than the degree μ of f_0; then the degree of g_{k+1} will be less than the degree ν of g_0. Since the induction is complete, our theorem is proved.

3. We readily deduce recursion formulas for the functions L_i. We have proved that functions L_i and ρ_i can be determined so that

$$f_i = bL_i - \rho_i f_0, \quad g_i = aL_i + \rho_i g_0 \qquad (i \geqq 1) \tag{12}$$

give functions f_i of degrees $< \mu$ and g_i of degrees $< \nu$ for which congruence (3) holds for every k. From (6),

$$af_0 + bg_0 = 1 + p\lambda(x). \tag{13}$$

In (10) we replace $F^{(k)}, f^{(k)}, g^{(k)}$ by the values obtained by replacing k by $k-1$ in (11) and then replace $F^{(k-1)}$ by the value obtained by replacing k by $k-1$ in (10). After deleting the factor p^k, we get

$$L_k = f_kg^{(k-1)} + g_kf^{(k-1)} + p^kf_kg_k + pL_{k+1} - pF_{k+1}.$$

In the terms $f_kg_0 + g_kf_0$ we replace f_k and g_k by their values

(12) and apply (13). After deleting the factor p we get

$$(14)\quad L_{k+1} = F_{k+1} - \lambda L_k - f_k[g]_{k-1} - g_k[f]_{k-1} - p^{k-1}f_k g_k,$$

in which we have employed the abbreviation

$$(15)\quad [z]_i = z_1 + pz_2 + p^2 z_3 + \cdots + p^{i-1}z_i, \quad [z]_0 = 0.$$

If in (10) we set

$$F^{(k)} = F_0 + p[F]_k, \quad f^{(k)} = f_0 + p[f]_k, \quad g^{(k)} = g_0 + p[g]_k,$$
$$\gamma = (F_0 - f_0 g_0)/p,$$

and then delete the factor p, we get

$$f_0[g]_k + g_0[f]_k + p^k L_{k+1} = \gamma + [F]_k + p^k F_{k+1} - p[f]_k[g]_k.$$

By (12), (15), the sum of the first two terms equals

$$\sum_{i=1}^{k} p^{i-1}(af_0 + bg_0)L_i = (1 + p\lambda)\sum_{i=1}^{k} p^{i-1}L_i = (1 + p\lambda)[L]_k.$$

Hence we obtain the formula

$$(16)\quad [L]_{k+1} = \gamma + [F]_{k+1} - p\lambda[L]_k - p[f]_k[g]_k.$$

It is also easy to establish this formula by induction.

4. To prove the more general theorem of § 1, we apply the method of § 2 with congruence (6) replaced by

$$(17)\qquad af_0 + bg_0 \equiv p^\rho \pmod{p^{\rho+1}}.$$

Since the resultant of $f_0(x)$ and $g_0(x)$ is divisible by p^ρ, but by no higher power of p, solutions $a(x)$ and $b(x)$ of (17) can be determined by the method given by Hensel on pages 62, 63. In view of (4),

$$F^{(\rho+1)} - f_0 g_0 = p^{\rho+1}L_1,$$

where L_1 is of degree less than n. Then the congruence

$$(18)\qquad F^{(\rho+1)} \equiv (f_0 + pf_1)(g_0 + pg_1) \pmod{p^{\rho+2}}$$

is satisfied if we take

$$f_1 \equiv p^{\sigma-\rho}(bL_1 - \rho_1 f_0), \quad g_1 \equiv p^{\sigma-\rho}(aL_1 + \rho_1 g_0) \pmod{p^{\sigma+1}},$$

since by (17)

$$(20)\qquad f_0 g_1 + f_1 g_0 \equiv p^\sigma L_1 \pmod{p^{\sigma+1}},$$

and $p^2 f_1 g_1$ is divisible by p^t, where $t = 2 + 2s - 2\rho > s + 1$. We choose $\rho_1(x)$ so that the degree of $f_1(x)$ shall be less than the degree of $f_0(x)$; then by (20) the degree of $g_1(x)$ will be less than the degree of $g_0(x)$.

Similarly, if in accord with (18) we set

$$F^{(s+1)} - (f_0 + pf_1)(g_0 + pg_1) = p^{s+2}(L_2 - F_{s+2}),$$

the congruence

$$F^{(s+2)} \equiv (f_0 + pf_1 + p^2 f_2)(g_0 + pg_1 + p^2 g_2) \pmod{p^{s+3}}$$

is satisfied if we take

$$f_2 \equiv p^{s-\rho}(bL_2 - \rho_2 f_0), \quad g_2 \equiv p^{s-\rho}(aL_2 + \rho_2 g_0) \pmod{p^{s+1}}.$$

The general step in the proof may now be made as in § 2.

HENSEL'S THEORY OF ALGEBRAIC NUMBERS.

Theorie der Algebraischen Zahlen. Von KURT HENSEL. Erster Band. Leipzig and Berlin, Teubner, 1908. xi + 346 pp.

IN the theory of functions one may investigate an analytic function in the neighborhood of a point $z = a$ by means of a power series in $z - a$. In arithmetic one usually employs only developments according to the fixed base 10. The author undertakes in the present work to introduce a corresponding mobility into arithmetic and algebra by employing expansions of numbers into power series in an arbitrary prime number p.

A positive integer D can be expressed in one and but one way in the form

$$D = d_0 + d_1 p + \cdots + d_k p^k,$$

in which each d_i is one of the integers 0, 1, $\cdots$, $p - 1$. This equation will be said to define the representation of D as a p-adic number, for which the following symbol will be employed:

$$D = d_0, d_1 d_2 \cdots d_k \ (p).$$

For example,

$$14 = 2 + 3 + 3^2 = 2{,}11 \ (3), \quad 216 = 2 \cdot 3^3 + 2 \cdot 3^4 = 0{,}0022 \ (3).$$

The sum of two such p-adic numbers is readily expressed as a p-adic number. For example,

$$0{,}0022 + 0{,}1021 = 0{,}10111 \ (3).$$

When $A \geqq B$, the difference $A - B$ is expressible as a p-adic number. Thus, if $A = 216$, $B = 138$, $p = 3$, we have

$$0{,}0022 - 0{,}1021 = 0{,}222 \quad (3).$$

But if $A < B$ a similar rule for subtraction (proceeding from left to right and borrowing when necessary unity from a later digit) would lead to an infinite sequence of digits. For example, if we take $p = 3$ and attempt to subtract, in this manner, 0,0022 from 0,1021, we obtain $0{,}10022\cdots$, in which the digit 2 is to be repeated indefinitely. It would be useless to define this symbol involving an infinitude of digits to be the infinite series

$$3 + 2\cdot 3^4 + 2\cdot 3^5 + \cdots + 2.3^n + \cdots,$$

which is divergent and not equal to 138–216. On the contrary we shall attach no numerical significance to such a symbol. Our procedure will be analogous to that employed in basing a theory of positive fractions upon the theory of positive integers by introducing symbols (a, b) involving a pair of integers. We here introduce symbols called p-adic numbers and define equality and the four rational operations. We shall prove that our set of p-adic numbers is closed under these operations and hence forms a field or domain. A certain subset of these p-adic numbers can be put into one-to-one correspondence with the rational numbers such that the sum, difference, product, or quotient of two p-adic numbers corresponds to the sum, etc., of the corresponding rational numbers. We shall then have a representation of each rational number as a p-adic number.

We note that, for $p = 3$, the quotient of $1 + 2\cdot 3 + 3^2$ by 3 is $3^{-1} + 2 + 3$, which is conveniently denoted by the symbol 12,1 (3).

Accordingly we introduce the symbols, called p-adic numbers,

$$D = d_{-\rho}\, d_{-\rho+1}\cdots d_{-1}\, d_0,\, d_1\, d_2\cdots \quad (p),$$

in which there is a finite number of *coefficients* d_i preceding the comma and a finite number or an infinitude of coefficients following the comma, while each d_i is a rational number in whose expression as a fraction in its lowest terms the denominator is not divisible by p. Such a fraction a/b is called *integral modulo* p since it plays the same rôle modulo p as the unique inte-

gral root of the congruence $bx \equiv a \pmod{p}$. A *reduced* p-adic number is one all of whose coefficients are integers of the set $0, 1, \cdots, p-1$.

For $k \geqq -\rho$, the rational number

$$D_k = d_{-\rho}p^{-\rho} + \cdots + d_{-1}p^{-1} + d_0 + d_1 p + \cdots + d_k p^k$$

is called the *convergent* D_k *of rank* k of the p-adic number D. For $k < -\rho$, we set $D_k = 0$. Thus if $D = 12{,}1$ (3), $D_{-1} = 3^{-1}$, $D_0 = 3^{-1} + 2$.

Two rational numbers will be called congruent modulo p^m if their difference equals the product of p^m by a number which is integral modulo p. Here m may be zero or any positive or negative integer. For example,

$$4 \cdot 5^{-2} \equiv \tfrac{2}{3} \cdot 5^{-2} \pmod{5^{-1}}, \quad 9 \cdot 5^{-2} \equiv \tfrac{2}{3} \cdot 5^{-2} \pmod{5^0},$$

since

$$4 \cdot 5^{-2} - \tfrac{2}{3} \cdot 5^{-2} = \tfrac{2}{3} \cdot 5^{-1}, \quad 9 \cdot 5^{-2} - \tfrac{2}{3} \cdot 5^{-2} = \tfrac{1}{3},$$

while $\frac{2}{3}$ and $\frac{1}{3}$ are integral modulo 5.

Two p-adic numbers D and D' are said to be *equal* when for every integer k their convergents D_k and D'_k of rank k are congruent modulo p^{k+1}. Note that $D_k \equiv D'_k \pmod{p^{k+1}}$ implies $D_l \equiv D'_l \pmod{p^{l+1}}$ for every $l < k$. For example,

$$D = p, p-1\, p-1\, p-1 \cdots, \quad D' = 0$$

are equal since

$$D_k = p + (p-1)p + \cdots + (p-1)p^k = p^{k+1}, \quad D'_k = 0.$$

In particular, two reduced p-adic numbers are equal if and only if their corresponding coefficients are identical. Every p-adic number equals a reduced p-adic number. The proof follows from the fact that any rational number which is integral modulo p can be expressed in the form $i + lp$, where i is one of the integers $0, 1, \cdots, p-1$, and l is integral modulo p. For example,

$$1\tfrac{2}{3},0 = 14{,}1313 \cdots (5)$$

with the repetend 13, since

$$\tfrac{2}{3} = 4 + (-\tfrac{2}{3})5, \quad -\tfrac{2}{3} = 1 + (-\tfrac{1}{3})5, \quad -\tfrac{1}{3} = 3 + (-\tfrac{2}{3})5.$$

If we prefix one or more zeros before a p-adic number D, we

obtain a p-adic number equal to D. Given any two p-adic numbers D and D', we may prefix enough zeros before one so that we obtain two numbers with the same number ρ of coefficients to the left of the comma. Then the *sum* of D and D' is defined to be the p-adic number

$$D + D' = d_{-\rho} + d'_{-\rho} \cdots d_{-1} + d'_{-1} \quad d_0 + d'_0, \; d_1 + d'_1 \cdots (p).$$

For example, $10{,}12 + 0{,}211 = 10{,}012$ (3). If $D = D_1$ and $D' = D'_1$, then $D + D'$ equals $D_1 + D'_1$.

The unique p-adic number X for which $X + D' = D$ is called the difference $D - D'$. Hence

$$D - D' = d_{-\rho} - d'_{-\rho} \cdots d_{-1} - d'_{-1} \quad d_0 - d'_0, \; d_1 - d'_1 \cdots (p).$$

For example, employing a bar to denote a repetend, we have

$$0{,}1\overline{2} - 0{,}2\overline{10} = 0{,}2\overline{01002} \quad (3).$$

The product P of two p-adic numbers D and D' is defined to be the p-adic number whose coefficients are those in the series obtained by the formal multiplication of the series

$$d_{-\rho}p^{-\rho} + \cdots + d_{-1}p^{-1} + d_0 + d_1 p + \cdots,$$

$$d'_{-\rho'}p^{-\rho'} + \cdots + d'_{-1}p^{-1} + d'_0 + d'_1 p + \cdots.$$

In particular, if D and D' are *integral* p-adic numbers,

$$D = d_0, \; d_1 d_2 \cdots, \quad D' = d'_0, \; d'_1 d'_2 \cdots (p),$$

their product is

$$P = d_0 d'_0, \quad d_0 d'_1 + d_1 d'_0 \quad d_0 d'_2 + d_1 d'_1 + d_2 d'_0 \cdots (p).$$

In general, $P = c_{-\gamma} c_{-\gamma+1} \cdots$, where $\gamma = \rho + \rho'$ and

$$c_{-\gamma} = d_{-\rho} d'_{-\rho'}, \quad c_{-\gamma+1} = d_{-\rho} d'_{-\rho'+1} + d_{-\rho+1} d'_{-\rho'}, \quad \cdots.$$

If the c's and d's are given numbers integral modulo p and d_ρ is not a multiple of p, the preceding equations uniquely determine each d' as a number integral modulo p, so that the quotient P/D is uniquely determined as a p-adic number $D' = d'_{\rho-\gamma} \cdots$. In particular, if D is a reduced p-adic number other than zero, P/D equals a p-adic number. For example,

$$3{,}12 \div 4{,}21 = 2{,}\overline{4220} \quad (5).$$

If $D = D_1$ and $D' = D'_1$, then $DD' = D_1 D'_1$ and $D/D' = D_1/D'_1$.

For addition and multiplication as defined above the commutative, associative and distributive laws are seen to hold. The totality of p-adic numbers forms a field or Körper $K(p)$, being closed under the above defined operations addition, subtraction, multiplication, and division (the divisor not being equal to zero).

Any rational number $\neq 0$ may be given the form

$$r = \frac{a}{b} \cdot p^n,$$

where n is zero or a positive or negative integer, while a and b are integers not divisible by p. To r we make correspond the *monomial* p-adic number all but one of whose coefficients are zero, that one being a/b and occupying the n-th place to the right of the comma if n is positive, and the $(1-n)$-th place to the left of the comma if n is zero or negative. To the rational number zero we make correspond the p-adic number zero. The product or quotient of two rational numbers r and r' obviously corresponds to the product or quotient of the corresponding monomial p-adic numbers. A like result is seen to hold for the sum or difference, if we make use of the fact that the monomial p-adic number with the coefficient $p^k d$ in the n-th place equals a monomial with the coefficient d in the $(k+n)$-th place. These monomial p-adic numbers when expressed as reduced p-adic numbers are periodic, and conversely any periodic p adic number equals a monomial (Hensel, page 38). Hence the set of all periodic reduced p-adic numbers is simply isomorphic with the domain of all rational numbers. The p-adic number, whether reduced or not, which corresponds to the rational number r will be designated $[r]$.

Any p-adic number $E = e_0, e_1 e_2 \cdots$, in which e_0 is not a multiple of p, is called a *unit* for the domain $K(p)$. Hence any p-adic number D can be expressed as a product $[p^n]E$. The exponent n is zero or a positive or negative integer and is called the *order* of D. The product or quotient of two units is a unit. The order of a product is the sum of the orders of the factors. A p-adic number $D = [p^n]E$ is called integral if its order n is positive or zero, and fractional if n is negative.

Consider an integral function of a variable x

$$f(x) = A_0 x^n + A_1 x^{n-1} + \cdots + A_n$$

with p-adic coefficients A_i. If $A_i^{(k)}$ is the convergent of rank k of A_i, the function with rational coefficients

$$f^{(k)}(x) = A_0^{(k)}x^n + \cdots + A_n^{(k)}$$

is called the convergent of rank k of $f(x)$. Two integral functions $f(x)$ and $g(x)$ with p-adic coefficients are called equal if, for every integer k, their convergents $f^{(k)}(x)$ and $g^{(k)}(x)$ are congruent modulo p^{k+1}, namely, if the coefficients of like powers of x are congruent.

An integral function is called reducible or irreducible in the domain $K(p)$ according as it is or is not equal to the product of two integral functions, each of degree $\geqq 1$, with p-adic coefficients.

Any integral function $f(x)$ can be expressed as a product of a p-adic number and a *primary* function

$$F(x) = [p^n]x^n + B_1x^{n-1} + \cdots + B_n$$

whose coefficients are integral p-adic numbers not all divisible by p, that of the highest power of x corresponding to a power of p. The product of two primary functions is primary. It is readily seen that $f(x)$ is reducible if and only if its primary component $F(x)$ is the product of two primary functions.

In case the discriminant $D(F)$ of $F(x)$ is zero, $F(x)$ and its derivative $F'(x)$ have a common factor which can be determined by Euclid's process. We may therefore restrict our attention to a primary function $F(x)$ whose discriminant is not zero and hence is of the form $[p^\delta]E$, where $\delta \geqq 0$. Then (Hensel, page 68) $F(x)$ is reducible if and only if its convergent $F^{(\delta)}(x)$ is reducible modulo $p^{\delta+1}$. A similar argument (page 71) shows that if

$$F(x) \equiv f(x)g(x) \pmod{p^{r+1}}$$

and $r + 1 > 2\rho$, where ρ is the order of the p-adic number defined by the resultant of $f(x)$ and $g(x)$, then $F(x)$ is reducible in $K(p)$. An important special case is the following. If

$$F(x) \equiv f(x)g(x) \pmod{p},$$

where f and g have no common factor modulo p, then $F(x)$ is reducible in $K(p)$. Since

$$x^{p-1} - 1 \equiv (x-1)(x-2)\cdots(x-\overline{p-1}) \pmod{p},$$

we conclude that $x^{p-1} - 1$ has $p - 1$ linear factors in $K(p)$, so that there exist $p - 1$ p-adic numbers which are $(p - 1)$-th roots of unity. They are all powers of a primitive $(p - 1)$-th root of unity, designated by ω. For example,

$$\omega = 2,22\cdots (3), \quad \omega = 3,46\cdots (7).$$

Any p-adic number can be expressed in the form

$$B = p^{\rho}\omega^{\beta}e,$$

where e is a principal unit $1, a_1a_2\cdots (p)$. If μ is not divisible by p, B is the μ-th power of a p-adic number if and only if ρ is divisible by μ and β is divisible by the greatest common divisor d of μ and $p - 1$ (page 87). In particular if p is odd, $\sqrt{B}$ is a p-adic number only when ρ and β are both even. For example, $\sqrt{2}$ is not a 3-adic number, since $2 = \omega e$; while $\sqrt{2}$ is a 7-adic number, since $2 = \omega^2 e$. For p an odd prime, B is the p-th power of a p-adic number if and only if ρ is divisible by p and e is of the form $1, 0e_2e_3\cdots (p)$.

A number β is called algebraic if it is the root of at least one equation

$$x^m + B_1x^{m-1} + \cdots + B_m = 0 \tag{1}$$

with rational coefficients. If $B_1,\cdots, B_m$ are integers, the root β is called an integral algebraic number. If $B_1,\cdots, B_m$ are rational numbers which are integral modulo p, β is called an algebraic number integral modulo p. If β and γ are algebraic numbers integral modulo p then $\beta + \gamma, \beta - \gamma$ and $\beta\gamma$ are algebraic numbers integral modulo p. The roots of an equation

$$x^r + \beta_1x^{r-1} + \cdots + \beta_r = 0,$$

whose coefficients are algebraic numbers integral modulo p, are algebraic numbers integral modulo p.

If in (1) each $B_i = b_i/p^{\rho}$ where b_i is integral modulo p, then $\gamma = p^{\rho}\beta$ is a root of

$$\gamma^m + b_1\gamma^{m-1} + p^{\rho}b_2\gamma^{m-2} + \cdots p^{\rho(m-1)}b_m = 0.$$

Hence every algebraic number β can be given the form γ/p^{ρ} where γ is an algebraic number integral modulo p, and ρ is an integer $\geqq 0$.

Let α be an algebraic number and let

$$F(x) = x^\lambda + a_1 x^{\lambda-1} + \cdots + a_\lambda = 0 \tag{2}$$

be the equation irreducible in the domain of rational numbers which has the root α. The totality of the rational functions of α with rational coefficients forms a field $K(\alpha)$ of *degree* λ. As shown in the theory of algebraic numbers, there exist λ integral algebraic numbers $\gamma_1, \cdots, \gamma_\lambda$ of $K(\alpha)$ such that every algebraic number γ of $K(\alpha)$ can be expressed in one and but one way in the form

$$\gamma = u_1\gamma_1 + u_2\gamma_2 + \cdots + u_\lambda\gamma_\lambda, \tag{3}$$

where each u_i is a rational number, while every integral algebraic number is of the form (3), where now each u_i is an integer. The numbers $\gamma_1, \cdots, \gamma_\lambda$ are said to form a fundamental system for $K(\alpha)$. Since $\beta + \gamma$ and $\beta\gamma$ are algebraic numbers integral modulo p when β and γ are, it follows that

$$v_1\gamma_1 + \cdots + v_\lambda\gamma_\lambda \tag{4}$$

is an algebraic number integral modulo p if $v_1, \cdots, v_\lambda$ are rational numbers integral modulo p. Conversely (Hensel, page 121), any algebraic number integral modulo p can be given the form (4).

An algebraic number β is said to be divisible by p^ρ if $\beta = p^\rho\gamma$, where γ is integral modulo p. If $\beta = b_1\gamma_1 + \cdots$, the conditions are $b_i \equiv 0 \pmod{p^\rho}$, for $i = 1, \cdots, \lambda$.

Two algebraic numbers are called congruent modulo p^ρ if $\beta - \beta'$ is divisible by p^ρ. The conditions are $b_i \equiv b_i' \pmod{p^\rho}$ for $i = 1, \cdots, \lambda$.

If β is integral modulo p, so that each b_i is a rational number integral modulo p, we can determine integers e_i of the set $0, 1, \cdots, p-1$ such that $b_i = e_i + pc_i$, where c_i is integral modulo p. Hence $\beta = \epsilon + p\gamma$, where

$$\epsilon = e_1\gamma_1 + \cdots + e_\lambda\gamma_\lambda, \quad \gamma = c_1\gamma_1 + \cdots + c_\lambda\gamma_\lambda.$$

A number of the type ϵ is called a reduced number of $K(\alpha)$. Hence any algebraic number integral modulo p is congruent modulo p to a reduced number.

Just as we introduced a field $K(p)$ of all p-adic numbers containing a sub-field simply isomorphic with the field of all rational numbers, so we shall now introduce a field containing

a sub-field simply isomorphic with $K(\alpha)$. To this end we employ the symbols

$$\delta = \delta_{-\rho} \cdots \delta_{-1}\delta_0, \delta_1\delta_2 \cdots \quad (p),$$

called p-adic algebraic numbers, in which each δ_i is an algebraic number of $K(\alpha)$ integral modulo p. When each δ_i is a reduced number ϵ_i, δ is called a reduced p-adic algebraic number. By the convergent of rank μ of δ is meant

$$\delta^{(\mu)} = \delta_{-\rho}p^{-\rho} + \cdots + \delta_{-1}p^{-1} + \delta_0 + \delta_1 p + \cdots + \delta_\mu p^\mu,$$

which is a number of the domain $K(\alpha)$. Two p-adic algebraic numbers δ and γ are called equal if, for every integer μ, their convergents of rank μ are congruent modulo $p^{\mu+1}$. In particular, if δ and γ are reduced, they are equal only when corresponding coefficients δ_i and γ_i are identical. In view of the preceding paragraph, any p-adic algebraic number β equals a reduced p-adic algebraic number.

Addition and multiplication of p-adic algebraic numbers is defined as for p-adic numbers. The totality of p-adic algebraic numbers based upon $K(\alpha)$ is seen to form a field $K(p, \alpha)$.

Any number $\beta \neq 0$ of $K(\alpha)$ can be expressed in one and but one way in the form $\beta = \gamma p^n$, where γ is a number of $K(\alpha)$ integral modulo p and not divisible by p, while n is zero or a positive or negative integer. To β we make correspond the monomial p-adic algebraic number all but one of whose coefficients are zero, that one being γ and occupying the nth place to the right of the comma if n is positive, and the $(1-n)$-th place to the left if n is zero or negative. When this monomial is expressed as a reduced p-adic algebraic number its coefficients $\epsilon_n, \epsilon_{n+1}, \cdots$, form a periodic series. This follows from the fact that in (3) each u_i is rational and hence is represented by a periodic p-adic number. Hence $K(\alpha)$ is simply isomorphic to the sub-field of $K(p, \alpha)$ composed of the periodic reduced p-adic algebraic numbers. The p-adic algebraic number, whether reduced or not, which corresponds to the number β of $K(\alpha)$ will be designated $[\beta]$.

In the p-adic algebraic number δ, each coefficient δ_i is a linear function of $\gamma_1, \cdots, \gamma_\lambda$. Hence the convergent $\delta^{(\mu)}$ equals $w_1^{(\mu)}\gamma_1 + \cdots + w_\lambda^{(\mu)}\gamma_\lambda$, in which $w_i^{(\mu)}$ is the convergent of order μ of a p-adic number u_i. Hence

$$\delta = u_1[\gamma_1] + \cdots + u_\lambda[\gamma_\lambda]. \tag{5}$$

Let the function (2) with the root α be expressed as a product

$$F(x) = F_1(x)\,F_2(x)\cdots F_t(x) \quad (p) \tag{6}$$

in which $F_i(x)$ has p-adic coefficients, is irreducible in $K(p)$ and is of degree λ_i. Let $\alpha_1 = [\alpha]$, $\alpha_2, \cdots, \alpha_{\lambda_1}$ be the roots of $F_1(x) = 0$.

Since the number γ_i of $K(\alpha)$ equals a rational function of α with rational coefficients, it follows from (5) that

$$\delta = \phi(\alpha_1) \quad (p),$$

where ϕ is a rational function with p-adic coefficients. Thus

$$\delta_j = \phi(\alpha_j) \qquad (j = 1, \cdots, \lambda_1)$$

are the roots of the equation

$$g_1(y) = \pi(y - \delta_j) = y^{\lambda_1} + b_1 y^{\lambda_1 - 1} + \cdots + b_\lambda = 0 \quad (p), \tag{7}$$

with p-adic coefficients. If $g_1(y)$ has a factor $f(y)$ irreducible in $K(p)$ which vanishes for $y = \delta_1$, then $f[\phi(x)] = 0$ has one root α_1 in common with the equation $F_1(x) = 0$ irreducible in $K(p)$ and hence has all the roots of the latter, so that $f(y)$ vanishes for each δ_i. Hence the various irreducible factors of $g_1(y)$ have the same roots and are therefore identical. Thus $g_1(y)$ is either irreducible or a power of an irreducible function. Multiplying $g_1(y)$ by a suitable power of p, we obtain

$$G_1(y) = B_0 y^{\lambda_1} + B_1 y^{\lambda_1 - 1} + \cdots + B_{\lambda_1} = 0 \quad (p), \tag{7'}$$

where the B_i are integral p-adic numbers not all divisible by p. It follows (Hensel, pages 74, 75) that B_0 and B_{λ_1} are not both divisible by p. Hence either δ or $1/\delta$ satisfies an equation with integral p-adic coefficients. A root of such an equation is said to be algebraically integral. A p-adic algebraic number ϵ is called an algebraic unit if both ϵ and $1/\epsilon$ are algebraically integral.

The product $\delta_1 \cdots \delta_{\lambda_1}$ is called the partial norm $n_1(\delta)$ of $\delta = \delta_1$ with respect to the factor $F_1(x)$. Since

$$n_1(\delta) = (-1)^{\lambda_1} B_{\lambda_1} / B_0$$

is an integral p-adic number only when B_0 is not divisible by p, we conclude that δ is algebraically integral if and only if $n_1(\delta)$ is an integral p-adic number. In particular, ϵ is an algebraic unit if and only if $n_1(\epsilon)$ and its reciprocal are integral

p-adic numbers, namely, if $n_1(\epsilon)$ is a p-adic unit E. The product or quotient of two algebraic units is therefore a unit.

If δ and β are p-adic algebraic numbers, such that δ/β is algebraically integral, δ is said to be divisible by β. The condition is that $n_1(\delta)/n_1(\beta)$ shall be an integral p-adic number and hence that the order of $n_1(\delta)$ shall be equal to or greater than the order of $n_1(\beta)$. In case these orders are equal, δ/β is an algebraic unit ϵ and δ and β are said to be equivalent.

The norms $n_1(\delta)$ of all algebraically integral numbers δ not units, are of the form p^dE, where $d > 0$. Let f_1 be the least integer d, and π_1 a number for which

$$n_1(\pi_1) = p^{f_1}E. \tag{8}$$

If π is another number whose norm has the order f_1, then $\pi = \pi_1\epsilon$ so that π and π_1 are equivalent. Every algebraically integral number δ not a unit is divisible by π_1 since $d \geqq f_1$. In particular, π_1 has no divisor other than itself and the units ϵ. If the product of two algebraically integral numbers is divisible by π_1, one of the factors is divisible by π_1. For, if neither δ nor δ' is divisible by π_1, each is a unit and hence $\delta\delta'$ is a unit and not divisible by π_1. Hence π_1 has the characteristic properties of a prime number, and all the primes in $K(p, \alpha)$ are equivalent. Since $n_1(p) = p^{\lambda_1}$, π_1 is a divisor of p.

If β is any number of $K(p, \alpha)$, $n_1(\beta) = p^bE$. Set

$$b = \rho_1 f_1 + f_0, \quad 0 \leqq f_0 < f_1.$$

Then $\beta' = \beta/\pi_1^{\rho_1}$ has the norm $p^{b-\rho_1 f_1}E' = p^{f_0}E'$. Hence $f_0 = 0$, so that β' is a unit ϵ. Hence every number β of $K(p, \alpha)$ is of the form $\beta = \epsilon\pi_1^{\rho_1}$. Here ρ_1 is called the order of β; it is the quotient of the order of $n_1(\beta)$ by f_1. In particular, $p = \epsilon\pi_1^{e_1}$, where $e_1 = \lambda_1/f_1$.

Two numbers β and β' of $K(p, \alpha)$ are called congruent modulo π_1^r if $\beta - \beta'$ is divisible by π_1^r. Hence if β is integral there is a reduced number $\epsilon^{(0)}$ such that

$$\beta \equiv \epsilon^{(0)} \pmod{\pi_1}, \quad \beta = \epsilon^{(0)} + \pi_1\beta^{(1)}.$$

Similarly, $\beta^{(1)} = \epsilon^{(1)} + \pi_1\beta^{(2)}$, etc. Hence

$$\beta \equiv \epsilon^{(0)} + \epsilon^{(1)}\pi_1 + \epsilon^{(2)}\pi_1^2 + \cdots + \epsilon^{(k)}\pi_1^{(k)} \pmod{\pi_1^{k+1}}.$$

Any number δ of $K(p, \alpha)$ is of the form $\pi_1^{\rho_1}\epsilon$. Taking $\beta = \epsilon$, we obtain the congruence

$$\delta \equiv \epsilon^{(\rho_1)}\pi_1^{\rho_1} + \epsilon^{(\rho_1+1)}\pi_1^{\rho_1+1} + \cdots + \epsilon^{(k)}\pi_1^{(k)} \pmod{\pi_1^{k+1}}. \tag{9}$$

We have $\pi_1^{e_1} = p\epsilon$ where ϵ is a unit of $K(p, \alpha)$. Thus

$$\pi_1 = p^{1/e_1}\epsilon_1,$$

where $\epsilon_1 = \epsilon^{1/e_1}$ is an algebraic unit not belonging in general to $K(p, \alpha)$. Extending our definition of equivalence, we write

$$\pi_1 \sim p^{1/e_1}. \tag{10}$$

If we replace α_1 by another root α_i of $F_1(x) = 0$, π_1 is replaced by a prime equivalent to p^{1/e_1} and hence to π_1. Hence in view of (9) we obtain symbolical developments of the conjugate numbers $\delta(\alpha_i)$ in power series in p^{1/e_1}. These are analogous to the power series in $(z - \alpha)^{1/e_1}$ for the roots of an algebraic equation in the neighborhood of a branch point $z = \alpha$ of order e_1.

For another factor $F_i(x)$ in (6) we have primes

$$\pi_i \sim p^{1/e_i},$$

where e_i is a divisor of the degree λ_i of F_i. In order to avoid speaking of different, but equivalent, prime numbers, we associate with the factor $F_i(x)$ a unique prime divisor $\mathfrak{p}_i$ and say that a p-adic algebraic number δ is divisible by $\mathfrak{p}_i^{\rho_i}$ and by no higher power of $\mathfrak{p}_i$ if $\delta = \epsilon\pi_i^{\rho_i}$, so that the development, analogous to (9), begins with $\pi_i^{\rho_i}$. Thus p has the distinct prime divisors $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$. Every number γ of $K(\alpha)$ is therefore divisible by definite powers of these prime divisors. Then $\gamma\gamma'$ and γ/γ' are divisible by exactly the powers $\rho_i + \rho_i'$ and $\rho_i - \rho_i'$ of $\mathfrak{p}_i$. A number γ is algebraically integral if and only if it contains no one of the prime divisors $\mathfrak{p}_i$ to a negative power.

If $x_1, \cdots, x_{\lambda_i}$ are the roots of $F_i(x) = 0$, then

$$\delta_j = \phi(x_j) \quad (j = 1, \cdots, \lambda_i)$$

are the roots of $g_i(y) = 0$, an equation analogous to (7). Thus

$$g(y) = g_1(y) \cdots g_t(y)$$

is a function whose constant term c_n differs at most in sign from the complete norm $n(\delta)$ which equals the product $n_1(\delta) \cdots n_t(\delta)$ of the partial norms of δ. Let now δ be an integral number of $K(\alpha)$. Then the coefficients of $g(y)$ are rational integers since its roots are $\phi(\beta)$ where β ranges over all the roots of (2). The highest power of p dividing $n(\delta)$ is p^d, where $d = \rho_1 f_1 + \cdots + \rho_t f_t$. But an integer c_n has only a finite number of prime factors. Hence an integral algebraic number γ has only a finite number of prime divisors.

In terms of these prime divisors, Hensel obtains (pages 214–237) an expression for the discriminant of $K(\alpha)$, that is, the discriminant of any fundamental system. Further he determines (pages 261–280) the conditions under which the discriminants of the various numbers of $K(\alpha)$ have a common unessential divisor. Hensel here makes an important advance in the theory of algebraic numbers. For other important results the reader is referred to the text itself.

In a book of the original character of the present one, some minor defects may be expected and excused. On page 16, A/B need not be, as stated, one of the integral p-adic numbers $c_0, c_1, \cdots$. At the middle of page 34 occurs an equation, although equality of two fractional p-adic numbers has not yet been defined. If A, B, C are integral or fractional p-adic numbers such that $A = BC$, and if A_k, B_k, C_k are their convergents of rank k, it is stated on page 36 that

$$B_k \cdot C_k \equiv A_k \pmod{p^{k+1}}.$$

While this is true for integral p-adic numbers, it is in general false for fractional p-adic numbers. For if B is of negative order $-b$, and C of order $-c$, and $b \geqq c$, we must employ the convergent of rank $k+b$ of C in order to reach all the terms of the convergent of rank k of A. The above congruence holds for the modulus p^{k+1-b}. On page 69, fourth line below (7), greater than p^ρ should be equal to or greater than p^ρ; the proof is however valid. On page 96, before (2), read among all equations with rational coefficients. On page 121, fifth line from bottom, $v_i - v_i^{(0)}$ should have the denominator p. On page 123, the context shows the meaning of the term algebraically divisible; in the paragraph following (2), ganze should be preceded by modulus p, while $\bar{u}_i = p^\delta \bar{u}_i$ contains a misprint. In the eleventh line on page 124, the final letter p should be $\gamma^{(\lambda)}$. In the fifth line on page 135, B_λ should be B_0. In the theorem on page 161, the term Bereich occurs in two senses; in the first instance it should be $K(p, \alpha)$, in the second, $K(p)$. On page 326, negative powers of p may occur in the p-adic development of B.

In the above exposition of the elements of Hensel's theory, I have avoided Hensel's notation $\Sigma_{i=-\rho}^{\infty} c_i p^i$ for a p-adic number, and have not identified the rational numbers with their corresponding p-adic representations. The terms greater than and less than as applied to p-adic numbers (Hensel, page 19) are not

in accord with the usage for real numbers, so that if the rational numbers a, b correspond to the p-adic numbers $[a]$, $[b]$, we may have $a > b$, $[a] < [b]$. In setting up this correspondence, I have introduced the term monomial p-adic number. On page 130, Hensel assumes that the equation for α is irreducible in $K(p)$. Although not stated explicitly, this assumption underlies §§ 3–7 of the same chapter. In the present account I have therefore avoided this assumption and proceeded at once with the general case; see (6) above.

In addition to the intrinsic interest attached to the new fields or domains introduced by Hensel, his theory has proved to be of such importance in the difficult problems relating to discriminants that it must be granted a permanent footing in the theory of algebraic numbers.

L. E. DICKSON.

UNIVERSITY OF CHICAGO,
May 9, 1910.

SHORTER NOTICES.

Factor Table for the First Ten Millions. By D. N. LEHMER. Washington, D. C., Carnegie Institution of Washington, 1909. xiv + 476 pp.

THE publication of Lehmer's factor table marks an event of the greatest importance in the science of higher arithmetic. The chief factor tables published hitherto are the following: For the first, second and third millions, Burckhardt (Paris, 1817, 1814, 1816); fourth, fifth and sixth millions, Glaisher (London, 1879, 1880, 1883); seventh, eighth and ninth millions, Dase and Rosenberg (Hamburg, 1862, 1863, 1865). Rosenberg's manuscript for the tenth million was presented by his widow to the Berlin Academy of Sciences, but has disappeared. Crelle's manuscript for the third, fourth, and fifth millions was turned over to the Berlin Academy but was found to be too inaccurate for publication. Kulik's manuscripts, placed in charge of the Vienna Royal Academy in 1867 (see Encyklopädie der Mathematischen Wissenschaften, volume I, page 951; *Wiener Berichte*, volume 53, page 460) purport to give the smallest factor of all numbers up to one hundred million which are not divisible by 2, 3, or 5. In Kulik's manuscript each prime not exceeding 163 is represented by a

single character (letter or digit), and the higher primes by two characters. Although his scheme effects a decided abbreviation, it increases greatly the difficulty of comparing his manuscript with other tables. Kulik's manuscript is not accurate enough for publication; his tenth million was translated into the ordinary notation by Lehmer and found to contain 226 erroneous entries. Lehmer gives a list of the 246 errors found in the tables by Burckhardt, Glaisher, Dase and Rosenberg for the first nine millions.

The publication of Lehmer's factor table is timely in view of the difficulty of obtaining copies of certain of the earlier tables. It is now practicable for each arithmetician to have at hand, ready for instant use, a factor table in a single volume extending to ten million.

However, the most important question in the comparison of two factor tables is their relative accuracy. When a computer uses a table giving the values of a continuous function, his result should be approximately correct; any considerable error may be laid at his own door. But when an arithmetician relies on a factor table for the primality of a given number, he is entirely dependent upon the accuracy of the table; the entry is either exactly right or wholly wrong, — there is no question of approximation. The independent verification that a proposed large number is actually prime usually entails great labor.

It is therefore in place to inquire into the grounds for the belief that Lehmer's factor table is more accurate than the earlier tables. The sheets (13 by 25 inches) of the corrected typewritten copy of the table were photographed on glass, reduced in size to 12 by 16 inches. Photographic proofs were then corrected by the author. The corrected photographs were transferred to zinc plates, on which any necessary corrections were made. It is believed that this reproduction by photography instead of by movable types has eliminated several sources of error in construction of the earlier tables. Mention should also be made of certain new devices employed in the actual construction of the present table, which tended towards increased accuracy. A certain modification enabled Lehmer to employ stencils only only one-fourth of the length of Glaisher's. Consequently, Lehmer was able to employ the stencil method throughout, whereas Glaisher was compelled to have resort to the less accurate "multiple method" for primes higher than 307. Again, Lehmer's stencil possessed a certain symmetry

which was utilized to check its accuracy. Finally, Lehmer employed the improved plan of glueing the successive pages top to bottom in a continuous sheet, which was mounted upon a long bench with a roller at each end.

However, the claim for the greater accuracy of Lehmer's table does not rest so much upon these improvements in its construction as upon the fact that he was in a position to eliminate errors from the proof sheets by noting discrepancies with the earlier tables, as detected by a comparison entry for entry, a comparison made at least five times.

In Kulik's table the rate of error was approximately 1 in 1000 entries; in the other tables used for comparison the rate was lower. There is about an even chance that two computers working independently and with a rate as high as Kulik's will both make an error in the same place in a table extending to ten million. The probability that the same error will occur is therefore negligible.

An error in the case of a composite number is less likely to arise and is of less importance than in the case of a prime, as it would be detected at once by the person using the table. Hence the value of the check afforded by a count of the primes given in a table and its comparison with the number computed by Bertelsen (*Acta Mathematica*, volume 17) by a modification of Meissel's method (*Mathematische Annalen*, volumes 2, 21, 25). In a letter to the reviewer, Lehmer states that the actual count of the primes (including unity) in the first ten millions was found to be 664,580, which agrees with Bertelsen's computed number (increased by one to count unity), and that he has finished about three-fifths of a separate table of the primes less than ten million. In the latter work he checks each successive thousand with Glaisher's count and each successive fifty thousand with Bertelsen's computed number. No discrepancies with the latter have as yet appeared, while the frequent differences with Glaisher's count are all due to the presence of detected errors in the tables from which Glaisher made his count.

While the wide circle of mathematicians and amateurs interested in the theory of numbers is already under the greatest obligations to the Carnegie Institution of Washington for its aid to Lehmer in the construction of his factor table and for its publication, it will be fully satisfied only upon the completion of this monumental project by the publication of a separate table of primes. L. E. DICKSON.

Analytische Geometrie des Punktpaares, des Kegelschnittes, und der Fläche zweiter Ordnung. Erster Teilband. By Dr. OTTO STAUDE. Leipzig and Berlin, Teubner, 1910. x + 548 pp., with 181 figures.

THIS volume is a continuation of the Analytische Geometrie des Punktes, der geraden Linie und der Ebene by the same author which was reviewed by Professor G. N. Bauer in the BULLETIN, volume 14, page 452 (June, 1908) and his concise summary applies equally well to the present volume. The treatment of the quadric surface is not completed, but is left for a third volume, to be devoted largely to intersections of surfaces of the second order. The bibliography and index will appear in the later volume.

The work aims to be a "Handbuch" and not simply "Vorlesungen." The author aims to leave no important facts concealed by attempting brevity, and as a result the reading of the book is apt to become tiresome because of detail. No effort is spared to make it convenient for ready reference. The table of contents, chapter heads, subheads, and important theorems are all so arranged as to make catch words stand out prominently. Important results are also frequently summed up in the form of tables, which are very convenient for comparison.

The book is divided into two parts of equal length. The first part deals with the point pair and the conic section. The second degree equation is treated in detail in cartesian, polar, parametric, and trilinear coordinates. Curves of the second class are treated as fully as those of the second order and line coordinates are used freely. Naturally there is little novel in this part. The classification of curves (as well as surfaces) according to "rank" is novel. A curve of the second order or class is of rank three if it has no double elements, of rank two if a single double element, and of rank one if a single infinity of double elements. Similarly an ordinary quadric surface is of rank four. This new use of the word rank does not seem necessary nor of particular advantage. The treatment of point and line involutions is of exceptional clearness. In treating class curves more use might be made of the principle of duality, and the Clebsch bordered determinant notation for the line equation of a conic could be used to advantage. The importance of the polar system with the conic as the locus of incident elements is properly emphasized. The section on confocal systems, introducing elliptic and parabolic

coordinates is also extremely well presented. In the chapter on trilinear coordinates the symbolic notation of Clebsch would be much briefer than the clumsy multiple summations, and fully as intelligible to most readers. Since the author writes down the simultaneous invariant of a conic and a line, it would seem worth while to point out that for variable coordinates u_1, u_2, u_3 this is merely the line equation of the conic.

It is somewhat disappointing that some of the features which make Salmon's Conic Sections a valued companion are not developed. There is nothing on invariants and covariants of systems of conics which are the subject of Salmon's most fruitful chapter. The theory of reciprocal polars is not very fully developed and such ideas as radius of curvature, evolutes, etc., are not introduced.

Part II. on surfaces is developed along similar lines. The most interesting features are probably the treatment of polar properties, the early introduction of line coordinates, and above all the clear discussion of the linear complexes arising. The next volume will doubtless contain much more of interest in space geometry.

D. D. LEIB.

Analytische Geometrie der Kegelschnitte. By W. DETTE. Leipzig, Teubner, 1909. vi + 232 pages, with 45 figures.

THIS admirable elementary text on conic sections is worthy of examination by any teacher of that subject. Both in arrangement and in treatment, there are a number of innovations. The book is divided into three parts: the first of 94 pages is devoted entirely to theory and the study of the general equation in the form $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$; the next 50 pages contain a classified list of over twelve hundred examples illustrating the text; the remaining part of the volume is devoted to answers to, and also suggestions for solving, the examples of part two.

In the text itself, the six chapter headings: the point, the right line, the ellipse, the parabola, the hyperbola, and the determination of a conic through points and lines, promise little different from the old line American text. But on the first page the author introduces the idea of relative "mass numbers" or magnitudes; that is, for any segment of a line AB, we say $AB = -BA$. The author calls AB and BA the relative "mass numbers," their absolute magnitude being the same. As soon

as coordinates are defined, locus problems are solved and the equation of the circle derived before the straight line is mentioned. Oblique axes and polar coordinates are introduced from the start and every theorem or even remark which applies to oblique as well as rectangular coordinates is starred so as to attract attention. Transformations of coordinates are introduced only as needed. In common with most German texts the idea of harmonic points and lines is introduced early and kept in the foreground.

The fundamental division of curves represented by equations of the second degree is based on whether $ac - b^2 \gtreqless 0$, although the equation of the ellipse is first derived from its focal property. In deriving the polar equation, p, the "parameter," is defined as the focal ordinate, hence taking the vertex as the origin we have the test $y^2 < 2px$ for the ellipse, $y^2 = 2px$ for the parabola, etc. The parabola is defined as an ellipse with one focus at infinity and all properties are derived from this definition, although it is later pointed out that it is equally well the limiting case of the hyperbola.

But perhaps the most striking feature in a book of such an elementary character is the extent to which the pole and polar idea is used. The entire theory of directrices, conjugate diameters, centers, tangents, and axes is based on this in a most attractive way. The "polarized" forms, i. e., $U_{11} = ax_1^2 + 2bx_1y_1 + cy_1^2 + 2dx_1 + 2ey_1 + f = 0$, $U_{12} = ax_1x_2 +$, etc., are regarded as fundamental quantities, and conditions are stated in terms of them; e. g., if $U_{11} \gtreqless 0$, the point P_1 is outside, on, or within the curve; if $U_{12}^2 - U_{11}U_{22} = 0$, the line P_1P_2 is a tangent; if $U_{12} = 0$, the line P_1P_2 is harmonically divided by its intersections with the conic, and so on. While this is not new, it certainly is used more extensively than in any other elementary text.

The last chapter in the book is devoted largely to the theorems of Plücker, Pascal, and Brianchon. It must be very evident to the reader that the few pages of text contain most of the essentials of conic sections, presented in a most modern fashion. The author states that his only reasons for treating each of the conics separately are aesthetic and pedagogic.

D. D. Leib.

Henri Poincaré. Biographie, Bibliographie analytique des Ecrits. Par ERNEST LEBON. Paris, Gauthier-Villars, 1909. viii + 80 pp.

THIS little volume is the first of a series which M. Lebon proposes to publish on the "Savants du Jour." The idea of giving to the public a biographical and bibliographical record concerning the various eminent scholars of the present day is novel and has much to recommend it. The reading of the present volume cannot but be an inspiration to the rising generation of mathematicians. The author, or more accurately the compiler (the book contains nothing from the pen of M. Lebon himself), has here brought together from various sources a series of articles and addresses relating to Poincaré. The contents are divided into seven sections relating respectively to biography, pure mathematics, analytic and celestial mechanics, mathematical physics, philosophy of science, necrology, and miscellaneous publications.

The section on Biography contains a part of the address delivered by M. Frédéric Masson, director of the Académie Française, at the time (January 28, 1909) when Poincaré was received as a member of the forty immortals. In fourteen pages we are given a comprehensive sketch of his life. Born at Nancy, April 29, 1854, he received his doctorate from the University of Paris in 1879. His rise to fame was meteoric. During the years 1879–1881 he published in the *Comptes Rendus* his epoch-making results on the uniformization of algebraic curves and the solution of linear differential equations which at the early age of 27 gave him a recognized position in the front rank of mathematicians. In 1885 he received from the Académie des Sciences the Prix Poncelet for the "ensemble de ses travaux mathématiques." The readers of the BULLETIN who wish to read in more detail of his life and work may be referred to a recent number of the *Popular Science Monthly* (September, 1909) which contains a translation of the address of M. Masson. This address is followed in the volume under review by a list covering four and a half pages of degrees, titles, prizes, membership in scientific bodies, etc.

The second Section is devoted to his work in pure mathematics. The text is an extract from the report by M. G. Rados on the Prix Bolyai (awarded to Poincaré by the Hungarian Academy of Sciences, April 18, 1905). This is followed by a list of publications in pure mathematics containing 146

titles. The text of the third Section is an extract from the address delivered by Professor G. H. Darwin (February 9, 1909) in presenting to Poincaré the gold medal of the Royal Astronomical Society of London. It is a brief review of his contributions to analytic and celestial mechanics. This is followed by a list of 85 titles of his publications in this field. The fourth Section opens with another extract from the report of M. Rados relating to Poincaré's contributions to mathematical physics. The list of his writings in this domain comprises 78 titles. The text of the fifth Section is a review by M. Emile Faguet of Poincaré's most recent book on the philosophy of science entitled "Science et Méthode." His writings in this field comprise 51 titles. His book "Science and Hypothesis" has been translated into five foreign languages. The Section devoted to necrology simply gives a list of 17 addresses and notices by Poincaré, on the life and work of Laguerre, Halphen, Tisserand, Weierstrass, Cornu, Berthelot, Kelvin, etc. The last section gives a list of 51 titles on miscellaneous subjects.

Aside from the inspiration to be derived from a reading of the text of this volume, the complete list of Poincaré's writings is valuable for reference. The volume also contains a fine heliogravure portrait of Poincaré.

J. W. YOUNG.

Des Notations Mathématiques, Énumération, Choix et Usage. Par DÉSIRE ANDRÉ. Paris, Gauthier-Villars, 1909. xviii + 501 pp.

THE present work is practically confined, as regards both subject matter and method of treatment, to the needs of teachers of secondary mathematics. The paucity of historical references and the prolix treatment of many of the subjects tend to detract from its usefulness as a work of reference, and its general method of treatment seems better suited to the wants of those who are interested in seeing self-evident things stated in a clear and attractive form than of those who are seeking abstruse information. The perusal of such a work during periods of relaxation may, however, tend to make matters of secondary importance appear more attractive to the mathematician and thus it may lead to a keener appreciation of the entire mathematical structure.

The number of historical references is not only small but some of those which are given are apt to mislead the reader.

For instance on pages 3 and 177 we are told that the Arabic number symbols were apparently derived from a set of broken lines or angles, equal in number to the units which the various symbols represent. In the first edition of Ball's Short History of Mathematics, page 147, this hypothesis as regards the origin of these symbols is presented, together with the following statement: "This conjecture is ingenious, but I am not aware of any historical basis for it." In the second edition of this history no reference seems to have been made to this "conjecture," and most of the other reliable histories pass it in silence. On the other hand no other theory as regards the origin of these universal symbols is mentioned in the book under review, and the unguarded reader would naturally infer that the theory here advanced had stronger historic support than any other.

As another instance, illustrating the author's tendency to pursue interesting conjectures rather than to be guided by tame historic facts, we may cite his treatment of the symbol = on page 95. After stating that this symbol is attributed to Recorde, 1557, and after referring to the fact that it might have been selected to represent equality because of the equality of the two parallel line segments, he considers the probability of its having been derived from the ancient astronomical sign for the equinoxes. As is well known, Recorde stated explicitly that he selected this symbol because "noe 2 thynges can be moare equalle" than two equal and parallel line segments. It seems strange that any one should try to find better reasons for such a choice than those given by the man who made it.

From what has been said it results that the present work is not free from grave faults. On the other hand it has unusual merits. The lengthy discussion of the symbols in common use and the numerous references to details will doubtless lead many readers to observe matters which they formerly passed unnoticed. The book also offers mathematical reading matter which requires no more effort than the ordinary novel, and should be of no less interest to those who are familiar with the notation of elementary mathematics. It is divided into three parts, of nearly equal lengths, under the headings: enumeration, choice, and usage, respectively.

The first part is preceded by a preliminary discourse of fourteen pages, which closes with a statement of the object of the work and a brief description of the different parts. This part is devoted to an enumeration and explanation of the mathe-

matical symbols in actual use and it closes with directions as regards editorial work. The object is to give an exposition of the contemporaneous mathematical notation, omitting what is very seldom used. In the second and third parts a number of rules are given as regards the choice and the use of the symbols. Although these rules may appear to the reader as platitudes, yet their formulation is not without interest. The subject of mathematical notation is so important that such efforts to give a clear expression of what ought to be strict rules in the choice and use of notation deserve serious attention, especially on the part of the beginner.

The work closes with an unusually complete table of contents covering eleven pages. The number of the different subjects is very large. The various chapter headings are as follows: Nombres entiers, fractions, quantités déterminées, nombres indéterminés, signes d'operations, signes de coordination, signes de fonctions, signes de relations, notations de la géométrie, signes de la géométrie analytique, mathématiques appliquées, signes de rédaction, netteté du signe, précision du signe, rappel des propriétés de l'objet, rappel des rapports entre les objets, choix des signes généraux, mesure des quantités, objets d'une seule sorte en nombre déterminé, objets d'une même sorte en nombre indéterminé, objets de deux sortes, correspondances entre deux sortes de signes, objets de plus de deux sortes, cas difficiles, écriture des expressions, expressions mal écrites, structure des expressions, expressions abrégées ou condensées, notations particulières, relations, systèmes d'équations, notations initiales des problèmes, mise en équations, direction des calculs, verifications.

G. A. MILLER.

Applied Mechanics for Engineers. By E. F. HANCOCK. The Macmillan Company, 1909. xii + 385 pp.

IN writing this text the author has boldly, if not wisely, taken the viewpoint of the engineer in choosing the material which he considers essential, as well as in the method of presenting it. He has clearly in mind the fact that in an undergraduate course in mechanics for technical schools the greatest difficulty encountered by the student lies in seeing the application of theory to practice. As stated in the preface, no new material in the matter of principles and theory is presented. However, in the matter of application of the fundamental prin-

ciples the author has purposely entered the field of the engineer for new and up-to-date material. Of the 292 problems (with few answers) given for solution by the student many are adapted from original sources in engineering journals, and in this connection many references are cited. In this method of building up a text there lies the danger of having the choice of material influenced by local conditions, evidences of which crop out in many places in the book. It could easily be criticized by the engineer whose problems are mainly those of static equilibrium, and he would say that insufficient stress is placed on his line of work and that the side of engineering dealing with dynamics is over-emphasized. Certainly the strong influence of the latter is evident. Again, in the choice of material the text might tend to become an engineer's hand book, which at best would be insufficient in content and poorly arranged for reference. In this connection it might seem quite superfluous to include in appendices tables of logarithms, trigonometric functions, squares, roots, etc., which the student has used in a more convenient form long before he studies mechanics.

In bringing into the text practical engineering problems the author handles his material well, and in adapting this material the danger of leaving vague and sometimes erroneous notions concerning engineering projects in the minds of the students is reduced to a minimum. To help present these problems, and in order to give clear notions of the theory the text is illustrated with 205 well-drawn figures, each of which tells its story well because it holds fast to fundamentals. The print is clear and the typographical errors noted are few and unimportant. Approximations and empirical formulas are used frequently because these often arise in engineering practice, and comparisons of results obtained from their use with theoretical values are made.

If a table of contents were given here it would show that the text contains chapters on only those fundamental notions which all would agree should be found in a text on mechanics. The author's claim for the completeness of the chapters on moment of inertia, center of gravity, work and energy, friction, and impact, both from the standpoint of theory and application, may be granted; at least for the applications.

The book contains considerable engineering data of value, which would be increased materially if the students made a judicious use of the references cited, a few of the more involved

theoretical discussions, and a maximum of application. It is plainly a text for students in technical courses, while its use in a more general course would hardly be recommended.

ERNEST W. PONZER.

NOTES.

THE July number (volume 11, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Die natürlichen Gleichungen der analytischen Curven in euclidischem Raume," by E. STUDY; "Two-dimensional chains and the associated collineations," by J. W. YOUNG; "Groups of rational transformations in a general field," by L. I. NEIKIRK; "On osculating element-bands associated with loci of surface-elements," by P. F. SMITH; "Fields of extremals in space," by G. A. BLISS, and M. MASON; "Groups generated by two operators s_1, s_2 satisfying the equation $s_1 s_2^2 = s_2 s_1^2$," by G. A. MILLER; "Congruences of the elliptic type," by L. P. EISENHART.

THE July number (volume 31, number 3) of the *American Journal of Mathematics* contains the following papers: "The osculants of plane rational quartic curves," by H. I. THOMSEN; "On the primitive groups of classes six and eight," by W. A. MANNING; "Minimalkurven als Oërter von Krümmungsmittelpunkten," and "Minimalcurven und Serret'sche Flächen," by E. STUDY; "On Steinerians of quartic surfaces," by J. N. VAN DER VRIES; "On the determination of the ternary modular groups," by R. L. BÖRGER; "Groups of transformations of Sylow subgroups," by G. A. MILLER.

THE concluding (July) number of volume 11 of the *Annals of Mathematics* contains: "The theory of shadow rails," by W. H. JACKSON; "On a new method of computing the roots of Bessel's functions," by W. MARSHALL; "A functional equation for the sine," by E. B. VAN VLECK; "Periodic decimal fractions," by W. H. JACKSON; "Concerning the invariant points of commutative collineations," by W. B. FITE; "A new construction for cycloids," by H. SCHAPPER; "Metric classification of conics and quadrics by means of rank," by G. RUTLEDGE; "A method of solving linear differential equations," by P. A. LAMBERT.

AT the public Leibniz session of the royal Prussian academy of sciences held in Berlin on June 30 it was announced that as no competent memoir had been received, the subject announced in 1905 for which the Steiner prize was offered (BULLETIN, volume 12, page 42) is now withdrawn. The prize, of 6000 marks, was awarded to Professor GASTON DARBOUX in recognition of his contributions to the theory of surfaces.

The academy proposed the following problem for the solution of which the Steiner prize of 7000 marks will be awarded in 1914:

"To determine all those non-degenerate surfaces of order five on which one or more series of conics lie, and to investigate their properties. It is required to confirm the correctness and completeness of the solution by furnishing an analytic commentary on the results of the geometric investigation."

The academy also announced the following problem for the solution of which the academy's prize of 5000 marks will be awarded:

"The number of classes in the general cyclotomic field is to be determined and compared with the number of classes of its divisors."

In both cases competing memoirs should be written in German, French, Latin, English, or Italian, and be submitted to the secretary of the academy under the usual conditions on or before December 31, 1913.

THE royal academy of sciences of Belgium announces the following prize problems for the year 1911:

1. Develop and systematize our knowledge of the physical constitution of the sun. Prize of 800 francs.

2. Extend our knowledge of the nature of osmotic pressure. Prize of 600 francs.

3. A new contribution to the development of functions (real or analytic) in series of polynomials. Prize of 800 francs.

4. A resumé of the memoirs on systems of conics in space, together with a new contribution concerning such systems. Prize of 800 francs.

THE following courses in mathematics are announced for the winter-semester 1910–1911.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Differential calculus, four hours; Applications of elliptic functions, four hours; Theory of complex numbers, two hours;

Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Algebra, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: General theory of functions of a complex variable, four hours; Theory of potential in the plane and in space, four hours; Seminar, two hours. — By Professor G. HETTNER: Infinite series, products, and continued fractions, two hours. — By Professor J. KNOBLAUCH: Determinants, four hours; Theory of surfaces and space curves, four hours; Mathematical exercises, one hour. — By Professor R. LEHMANN-FILHÉS: Analytic geometry, four hours; Planetary orbits, four hours.

UNIVERSITY OF BONN. — By Professor E. STUDY: Analytic mechanics, four hours; Seminar, two hours. — By Professor F. LONDON: Analytic geometry of the plane and of space, four hours; Descriptive geometry, three hours; Seminar, two hours. — By Professor F. HAUSDORFF: Differential and integral calculus, II, four hours; Introduction to the theory of groups, two hours. — By Dr. J. MÜLLER: Theory of potential, three hours.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Development of mathematics in the nineteenth century, four hours; Seminar, two hours. — By Professor D. HILBERT: Mechanics, four hours; Seminar, two hours. — By Professor E. LANDAU: Differential and integral calculus, five hours; Seminar, two hours. — By Professor C. RUNGE: Descriptive geometry, eight hours; Seminar, two hours. — By Professor L. PRANDTL: Aerial mechanics, four hours; Seminar, two hours. — By Dr. P. KOEBE: Determinants and applications, two hours; Introduction to the theory of functions, six hours. — By Dr. O. TOEPLITZ: Algebra, four hours; Theory of sets in elementary presentation, two hours. — By Dr. F. BERNSTEIN: Elliptic functions, four hours; Calculus of insurance, two hours. — By Dr. C. MÜLLER: History of the discovery of the calculus, two hours. — By Dr. A. HAAR: Introduction to the calculus of variations, two hours; Theory of perturbations, two hours. — By Dr. H. WEYL: Developments in series in mathematical physics, four hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Selected chapters of mathematics, three hours. — By Professor O. HÖLDER: Analytic mechanics, five hours; Theory of numbers, two hours; Seminar, two hours. — By Professor K. ROHN:

Determinants, two hours; Applications of the calculus to curves and surfaces, four hours; Seminar, two hours. — By Professor G. HERGLOTZ: Differential and integral calculus, five hours; Elements of the theory of partial differential equations, two hours; Seminar, two hours. — By Professor H. LIEBMANN: Analytic geometry of space, five hours; Non-euclidean geometry, two hours; Seminar, two hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Theory of functions of a complex variable, four hours; Plane analytic geometry, four hours; Introduction to the theory of groups of transformations, two hours; Seminar, two hours. — By Professor R. v. SEELIGER: Celestial mechanics, four hours. — By Professor A. VOSS: Algebra, four hours; Theory of differential equations, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Differential calculus, five hours; Definite integrals and Fourier's series, four hours. — By Professor A. SOMMERFELD: Analytic mechanics, four hours; Geometric optics, two hours. — By Professor L. BRUNN: Recent researches in analysis situs, two hours. — By Professor K. DOEHLEMANN: Descriptive geometry with exercises, eight hours; Line geometry synthetically treated, four hours; The imaginary in geometry, one hour. — By Dr. G. HARTOGS: Theory of space curves and surfaces, four hours.

UNIVERSITY OF STRASSBURG. — By Professor H. WEBER: Differential and integral calculus, four hours; Elliptic functions, two hours; Seminar, two hours. — By Professor F. SCHUR: Projective geometry of space, four hours; Foundations of geometry, two hours; Seminar, two hours. — By Professor J. WELLSTEIN: Introduction to the theory of algebraic functions, four hours; Riemann surfaces, two hours; Seminar, two hours. — By Professor L. v. MISES: Analytic geometry, four hours; Kinematics of rigid bodies, two hours; Seminar, two hours. — By Professor M. SIMON: History of mathematics, two hours. — By Professor P. EPSTEIN: Introduction to the theory of linear differential equations, two hours.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1910–1911. Courses in algebra, analytic geometry, projective and descriptive geometry, and elementary courses in the calculus are not included:

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELÀ: Lebesgue's integrals; advanced mechanics, three hours. — By Professor L. DONATI: Electromagnetic field, principle of relativity, three hours. — By Professor L. PINCHERLE: Distributive operations, integral and differential equations (especially of the second order, in both complex and real field), three hours.

UNIVERSITY OF CATANIA. — By Professor M. DE FRANCHIS: Differential geometry with applications to non-euclidean geometry, four hours. — By Professor L. LAURICELLA: Mathematical theory of elasticity with applications, four hours. — By Professor G. PENNACCHIETTI: Principles of celestial mechanics, with applications, four hours. — By Professor C. SEVERINI: General theory of functions, four hours.

UNIVERSITY OF GENOA. — By Professor E. E. LEVI: Foundations of the theory of functions of a real variable; calculus of variations, three hours. — By Professor G. LORIA: Theory of groups of transformations, three hours. — By Professor O. TEDONE: Special problems on equilibrium and motion of elastic bodies, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO: History of mathematics before 1200, three hours. — By Professor R. MARCOLONGO: Vector analysis and its applications to hydromechanics, elasticity, and electro-mechanics, three hours. — By Professor D. MONTESANO: Birational correspondences in space; geometry of straight lines and of conics as elements of space, four and one half hours. — By Professor E. PASCAL: Linear differential equations, and the theory of groups of transformations, three hours. — By Professor E. PINTO: Physical optics with special regard to the phenomena of diffraction, four and a half hours. — By Professor G. TORELLI: Analytic theory of numbers, four and a half hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS: General theory of functions, elliptic functions, four hours. — By Professor U. CISOTTI: Mathematical theory of elasticity with technical applications, three hours. — By Professor A. FARARO: The teaching of mathematics in the University of Padua from the fourteenth to the seventeenth century, three hours. — By Professor P. GAZZANIGA: Theory of numbers, three hours. — By Professor T. LEVI-CIVITA: Statistical mechanics; kinetic theory of gases, four and one half hours. —

By Professor G. RICCI: Absolute differential calculus; harmonic and poli-harmonic functions; general methods in elasticity, four hours. — By Professor F. SEVERI: Theory of algebraic functions of two variables and of their integrals, four hours.— By Professor G. VERONESE: Foundations of geometry, four hours.

UNIVERSITY OF PALERMO. — By Professor G. BAGNERA: Partial differential equations of the second order, three hours. — By Professor M. GEBBIA: Vibrations of elastic solid media; applications to acoustics and optics, four and one half hours.— By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one half hours. — By Professor A. VENTURI: Planetary motion about the sun and about the center of mass; forms of planets, four and one half hours.

UNIVERSITY OF PAVIA.— By Professor E. ALMANSI: Theory of heat, three hours.— By Professor L. BERZOLARI: Geometry on an algebraic curve, three hours. — By Professor F. GERBALDI: Elliptic functions, three hours. — By Professor G. VIVANTI: General theory of functions with application to integral functions, three hours.

UNIVERSITY OF PISA.— By Professor E. BERTINI: Fundamental properties of the geometry on an algebraic surface, three hours.— By Professor L. BIANCHI: Preliminaries on ordinary and partial differential equations; infinitesimal geometry of curves and surfaces, four and one half hours. — By Professor U. DINI: Functions of a complex variable, elliptic functions, four and one half hours.— By Professor G. A. MAGGI: Equations of dynamics, potential and harmonic functions, vector fields with applications, four and one half hours.— By Professor P. PIZZETTI: Interpolation and numerical integration; notions in spherical astronomy, figure of the planets, three hours.

UNIVERSITY OF ROME. — By Professor G. BISCONCINI: Differential geometry in its connection to mechanics, three hours. — By Professor G. CASTELNUOVO: Foundations of geometry, non-euclidean geometry, three hours.— By Professor L. ORLANDO: Analytical foundation of mathematical physics, three hours. — By Professor V. VOLTERRA: General types of differential equations of mathematical physics and of celestial mechanics, three hours. — By ————: Higher analysis, three hours.

UNIVERSITY OF TURIN. — By Professor T. BOGGIO: Integral equations and theory of potential, three hours. — By Professor G. SANNIA: Non-euclidean geometry, three hours. — By Professor C. SEGRE: Geometry of birational transformations of curves and surfaces, three hours.—By Professor C. SOMIGLIANA: Theory of potential with applications, three hours. — By ——— ———: Higher analysis, three hours.

THE following mathematicians have been elected corresponding members of the royal academy of sciences of Turin: Professor J. BOUSSINESQ, of the University of Paris, Professor E. CAVALLI, of the University of Naples, Professor V. CERULLI, of Teramo, Italy, Professor Sir G. H. DARWIN, of Cambridge University, Professor F. ENRIQUES, of the University of Bologna, Professor G. P. GUCCIA, of the University of Palermo, Professor T. LEVI-CIVITA, of the University of Padua, Professor C. NEUMANN, of the University of Leipzig. Professor M. NOETHER, of the University of Erlangen, has been elected foreign member.

AT the University College of London, Mr. F. JACKSON and Miss M. PICK have been appointed associate professors of mathematics. Mr. P. F. EVERITT has been appointed honorary associate professor of mathematics.

DR. M. PLANCHEREL has been appointed docent in mathematics at the University of Geneva.

DR. L. GABEREL, of the University of Neuchâtel, has been promoted to an associate professorship of mathematical analysis.

DR. P. FURTWÄNGLER, of the technical school at Aachen, has been appointed professor of mathematics at the agricultural institute of Bonn-Poppelsdorf.

DR. L. LICHTENSTEIN has been appointed docent in mathematics at the technical school of Berlin.

DR. E. FISHER has been promoted to an associate professorship of mathematics at the technical school of Brünn.

PROFESSOR F. HASENÖHRL has been elected corresponding member of the academy of sciences of Vienna.

PROFESSOR B. HOPKINSON has been elected a member of the royal society of London.

PROFESSOR K. VON DER MÜHLL has received the honorary degree of doctor of laws from the University of Basel.

PROFESSOR H. A. SCHWARZ, of the University of Berlin, has been decorated with the order of the red eagle of the third class with shield.

DR. C. MÜLLER, of the University of Göttingen, has been appointed professor of mathematics at the technical school of Hannover.

PROFESSOR E. STEINITZ, of the technical school at Berlin, has been appointed professor of higher mathematics at the technical school of Breslau.

DR. TH. V. KÁRMÁN has been appointed docent in mechanics at the University of Göttingen.

PROFESSOR G. LAURICELLA, of the University of Catania, has been appointed professor of mathematics at the University of Rome, as successor to the late Professor CERRUTI.

PROFESSORS E. CIANI, of the University of Genoa, B. LEVI, of the University of Cagliari, and G. FUBINI, of the technical school of Turin, have been promoted to full professorships of mathematics.

DR. L. SILLA has been appointed docent in rational mechanics at the University of Rome.

DR. C. AIMONETTI has been appointed docent in geodesy at the University of Turin.

DR. G. FORNI has been appointed docent in geodesy at the technical school of Milan.

DR. W. A. GRANVILLE, of Yale University, has been elected president of Pennsylvania College, Gettysburg, Pa.

DR. R. P. BAKER, of the University of Iowa, has been promoted to an assistant professorship of mathematics.

MISS M. B. WHITE has been appointed assistant professor of mathematics at the University of Kansas. Mr. G. W. HESS has been appointed instructor in mathematics.

PROFESSOR OSWALD VEBLEN, of Princeton University, has been promoted to a full professorship of mathematics.

PROFESSORS J. B. SHAW, of the James Milliken University, and ARNOLD EMCH, of the cantonal school of Soloturn, Switzerland, have been appointed assistant professors of mathematics at the University of Illinois.

THE following appointments to instructorships in mathematics are announced: Dr. C. D. BROOKS, at Northwestern University; Mr. B. E. CARTER, at Colby College; Mr. E. E. MOOTS, at the University of Wisconsin; Mr. C. J. WEST, at the University of Ohio; Mr. R. S. POND, at the University of Georgia.

PROFESSOR J. WEINGARTEN, of the University of Freiburg, died June 16, at the age of 74 years.

PROFESSOR J. V. SCHIAPARELLI, former director of the astronomical observatory at Milan, died July 4, at the age of 75 years.

PROFESSOR L. RAFFY, of the University of Paris, died June 9, at the age of 55 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

EPSTEIN (P.). See PASCAL (E.).

FUCHS (C. A.) und LUDWIG (F. K.). Zur Geometrie zweier und mehrerer Kreise. Komotau, Benker, 1910. 8vo. 54 pp. M. 1.30

HIME (H. W. L.). Anharmonic coordinates. London, Longmans, 1910. 12mo. 7s. 6d.

LUDWIG (F. K.). See FUCHS (C. A.).

MONTEIL (P. L.). Théorie du point (2e partie). Géométrie curviligne. 2e édition, revue, corrigée et considérablement simplifiée. Paris, Chapelot, 1910. 4to. 82 pp.

MÜLLER (E.). See SCHRÖDER (E.).

NETTO (E.). Die Determinanten. Leipzig, Teubner, 1910. 8vo. 6 + 129 pp. Cloth. M. 3.60

PASCAL (E.). Repertorium der höheren Mathematik. 2te völlig umgearbeitete Auflage der deutschen Ausgabe, herausgegeben von H. E. Timerding und P. Epstein. Band II, erste Hälfte: Grundlagen und ebene Geometrie. Leipzig, Teubner, 1910. 8vo. 16 + 534 pp. M. 10.00

SALMOIRAGHI (A.). Guida pratica della geometria moderna. Milano, Allegretti, 1910. 8vo. 24 + 120 pp. L. 5.00

SCHIMANSKI (E.). Die algebraischen Invarianten der projektiven Gruppen der Ebene und die geometrische Charakterisierung dieser Gruppen. (Diss.) Königsberg, 1910. 8vo. 95 pp.

SCHRÖDER (E.). Abriss der Algebra der Logik. Bearbeitet im Auftrag der deutschen Mathematiker-Vereinigung. Von E. Müller. Teil II: Aussagentheorie, Funktionen, Gleichungen und Ungleichungen. Leipzig, Teubner, 1910. 8vo. 6 + 51–159 pp. M. 4.00

SCHRÖDER (J.). Aufgaben für den Unterricht in der analytischen Geometrie der Ebene an höheren Schulen. Leipzig, Teubner, 1910. 8vo. 4 + 49 pp. Cloth. M. 1.40

SIGNORINI (A.). La transformazione Bk delle superficie applicabili sulle quadriche dello spazio ellittico. Pisa, Nistri, 1909. 8vo. 124 pp.

SONNET (H.) y FRONTERA (G.). Elementos de geometría analitica. Madrid, Bailly-Baillière, 1910. 8vo. 682 pp. P. 8.00

STOCKHAUS (H.). Beitrag zum Beweis des Fermatschen Satzes. Leipzig, Brandstetter, 1910. 8vo. 7 + 90 pp. M. 2.00

TIMERDING (H. E.). See PASCAL (E.).

II. ELEMENTARY MATHEMATICS.

BELLACCHI (G.). Complementi di geometria e d'algebra. Firenze, Barbèra, 1910. 16mo. 161 pp. L. 3.00

BERSANO (G. B.). Geometria per uso delle scuole tecniche. 2a edizione, corretta. Torino, Olivero, 1910. 8vo. 186 pp. L. 2.60

BOUGUERET (A.). Cours de dessin géométrique à l'usage des enseignements secondaire, primaire et professionnel. 2me partie: Projections; Eléments; Rotation des solides; Sections planes. Paris, Thomas, 1910. 4to. 10 pp.

——. Cours de dessin géométrique à l'usage des enseignements secondaire, primaire et professionnel. 6ème partie: Architecture. Paris, Thomas, 1910. 4to. 10 pp.

BRIGGS (W.). Second stage mathematics, with modern geometry. London, Clive, 1910. 12mo. 3s. 6d.

D'OVIDIO (E.). See SANNIA (A.).

DRUXES (J.). Ausführlicher Lehrgang der Arithmetik und Algebra nach modernen Grundsätzen. Für Lehrer und Studierende. Band I. Cöln, M. du Mont-Schauberg, 1910. 8vo. 7 + 445 pp. M. 8.00

DURELL (C. V.). A course in plane geometry for advanced students. Part II. New York, Macmillan, 1910. 8vo. 14 + 358 pp. Cloth. $2.40

EBNETER (K.). Aufgaben der elementaren Algebra. Methodisch geordnete Sammlung für die 3te Klasse an Sekundar- oder Realschulen, sowie für Gewerbeschulen und verwandte Lehranstalten. St. Gallen, Fehr, 1910. 8vo. 6 + 67 pp. M. 0.80

FORNIER Y PADILLA (F.). Nociones de aritmética y geometría. Barcelona, Carbonell, 1910. 214 pp. P. 2.00

FÜRLE (H.). Ein Rechenblatt zur Auflösung der Gleichung vierten Grades mit Hilfe des Zirkels. (Progr.) Berlin, Weidmann, 1910. 8vo. 16 pp. M. 1.00

GOLLER (J.). Raumlehre mit geometrischem Zeichnen, bearbeitet im Auftrage des württembergischen evangelischen Lehrerunterstützungsvereins. Schülerausgabe A, für die Volkschule. 2te Auflage. Stuttgart, Bonn. 1910. 8vo. 48 pp. M. 0.40

HOLLMANN (T.). Mathematik für höhere Mädchenschulen. Nach den neuen Lehrplänen bearbeitet. II. Arithmetik und Algebra. Düsseldorf, Schwann, 1910. 8vo. 6 + 175 pp. Cloth. M. 2.50

Lösungen. 36 pp. M. 0.80

JAQUET (E.) et LACLEF (A.) Eléments d'algèbre. Exercices et problèmes à l'usage des écoles primaires supérieures, des écoles professionnelles, des écoles normales d'instituteurs. Nouvelle édition. Paris, Nathan, 1910. 16mo. 190 pp.

LAYNG (A. E.). A general text-book of elementary Algebra. Books 1 and 2. London, Blackie, 1910. 12mo. 2s. 6d.

MARVÁ Y MEYER (J.). See RODRÍGUEZ PERLADO (P.).

MOORE (E. H.). See MYERS (G. W.).

MÜLLER (H.) und MAHLERT (A.). Mathematisches Lehr- und Uebungsbuch für höhere Mädchenschulen. Teil I: Arithmetik und Algebra. 3te Auflage. Leipzig, Teubner, 1910. 8vo. 6 + 146 pp. Cloth M. 2.00

MYERS (G. W.). Second-year mathematics for secondary schools; edited by E. H. Moore. Chicago, University of Chicago Press, 1910. 14 + 282 pp. Cloth. $1.50

NAMPON (G.). Cent questions de théorie. Arithmétique, algèbre, géométrie des examens du brevet supérieur avec développements et solutions. 2me édition, revue. Paris, Hachette, 1910. 16mo. 111 pp. Fr. 1.00

OBERG (E.). Use of logarithms and logarithmic tables. With tables of logarithms. New York, Industrial Press, 1910. 8vo. 35 pp. $0.25

RODRÍGUEZ PERLADO (P.). Tratado de geometría elemental. Con prólogo de J. Marvá y Meyer. Madrid, Hernando, 1910. 391 pp. P. 7.50

SALAZAR é IBÁÑEZ (M.). Problemas matemáticas. Contestaciones al programa de esta asignatura, para las oposiciones al Cuerpo de Aduanas. Madrid, Marzo, 1910. 8vo. 300 pp. P. 7.00

SANNIA (A.) e D'OVIDIO (E.). Elementi di geometria. Vol. I, ad uso dei ginnasii e degli istituti technici. 13a edizione, riveduta e corretta. Napoli, Pellerano, 1910. 16mo. 15 + 204 pp. L. 2.00

SCHÜLKE (A.). Aufgaben-Sammlung aus der reinen und angewandten Mathematik. 2ter Teil, für die oberen Klassen höheren Schulen. 2te Auflage. Leipzig, Teubner, 1910. 8vo. 10 + 186 pp. Cloth. M. 2.40

SCHULTZE (A.). Elements of Algebra. New York, Macmillan, 1910. 12mo. 12 + 309 pp. Cloth. $0.85

WEBER (H.) und WELLSTEIN (J.). Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. Band I: Encyklopädie der elementaren Algebra und Analysis. Bearbeitet von H. Weber. 3te Auflage. Leipzig, Teubner, 1910. 8vo. 18 + 532 pp. Cloth. M. 10.00

WEBER (K.). Lehrbuch der Planimetrie für höhere Schulen und zum Selbstunterricht. Wolfenbüttel, Zwissler, 1910. 8vo. 164 pp. M. 3.00

WELLSTEIN (J.). See WEBER (H.).

WINKLER. Lehrbuch der Geometrie für höhere Mädchenschulen. Auf Grund der ministeriellen Bestimmung über die Neuordnung des höheren Mädchenschulwesens vom 8. VIII und 12. XII. 1908 nach Spieker's "Lehrbuch der Geometrie für höhere Lehranstalten" bearbeitet. Teil 3, Klasse II der höheren Mädchenschule. Potsdam, Stein, 1910. 8vo. 36 pp. M. 0.50

III. APPLIED MATHEMATICS.

ALBE (E. E. F. D'). The electron theory; a popular introduction to the new theory of electricity and magnetism; with a preface by G. J. Stoney. 3rd edition. New York, Longmans, 1910. 12mo. 27 + 327 pp. Cloth. $1.25

BAZARD (A.). Cours de mécanique. Vol. IV: Chaudières et machines à vapeur. Turbines à vapeur. Paris, Geisler, 1910. 8vo. 593 pp. Fr. 12.00

BEAUFILS (H.). See HARANG (F.).

BORCHARDT (W. G.). Key to elementary statics. London, Rivingtons, 1910. 12mo. 10s. 6d.

BUROK (E.). See HESS (L.).

CAUNT (G. W.). See JESSOP (C. M.).

CHOLLET (T.) et MINEUR (P.). Traité de géométrie descriptive. 7e édition. Paris, Vuibert et Nony, 1910. 8vo. 6 + 336 pp.

DREYER (G.). Elemente der Graphostatik. Lehrbuch für technische Unterrichtsanstalten und zum Selbstunterricht, mit vielen Anwendungen auf den Maschinenbau und Brückenbau. 3te Auflage. Ilmenau, Reinmann, 1910. 8vo. 6 + 121 pp. Cloth. M. 7.50

DUNCAN (R. H.). Practical curve tracing. With chapters on differentiation and integration. New York, Longmans, 1910. 8vo. 8 + 138 pp. Cloth. $1.60

FARJON. See GOUARD (E.).

FERVAL (H.). Eléments de géométrie descriptive à l'usage des candidats aux baccalauréats de l'enseignement secondaire et aux écoles du gouverne ment. 10me édition, conforme aux programmes du 27 juillet 1905. Paris, Belin, 1910. 12mo. 350 pp. Fr. 3.75

FISCHER (M.). Statik und Festigkeitslehre. Vollständiger Lehrgang zum Selbststudium für Ingenieure, Techniker und Studierende. 1ter Band: Grundlagen der Statik und Berechnung vollwandiger Systeme einschliesslich Eisenbeton. 2te, vermehrte Auflage. Mit zahlreichen Beispielen und Zeichnungen. Berlin, Meusser, 1910. 8vo. 10 + 645 pp. Cloth. M. 18.00

FUCINI (C.). Lezioni di geometria descrittiva. Genova, Gioventù, 1910. 8vo. 80 pp. L. 1.80

GOUARD (E.) et HIERNAUX (G.). Cours élémentaire de mécanique industrielle. Principes généraux, applications, exercices pratiques. Préface de M. Farjon. A l'usage des écoles pratiques de commerce et d'industrie (rédigé conformément aux programmes du 28 août 1909), des écoles nationales professionnelles, des écoles des mécaniciens de la marine, etc. Vol. I. Paris, Dunod, 1910. 8vo. 16 + 328 pp. Fr. 4.00

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